Markets' Notion on Implied Volatility Risks: Insights from Model-Free VIX Futures Pricing

Hendrik Hülsbusch* Alexander Kraftschik*

This version: May 5, 2017

Abstract

This paper studies the interdependencies between the VIX futures market and the S&P500 and VIX options markets using a model-free pricing method for VIX futures. We show that the replication strategy for the VIX futures deviates strongly from observed prices. Limited strike ranges do not suffice to reason these deviations, whereas liquidity risks can explain most of it. After controlling for liquidity by constructing higher and lower bounds for the VIX futures price, we find a lead-lag structure between markets segmented by product, not by its underlying. Our model-free analysis shows that if option markets imply higher volatility risks relative to VIX futures, option prices in both markets adjust and vice versa.

Keywords: Model-Free Pricing, S&P500 Options, VIX Futures, VIX Options, Volatility Risk

JEL: G13, G14

* School of Business and Economics, Finance Center Muenster, University of Muenster, Universitätsstr. 14-16, 48143 Münster, Germany. E-mail: hendrik.huelsbusch@wiwi.uni-muenster.de, alexander.kraftschik@wiwi.uni-muenster.de

We thank Friedrich Lorenz, Michael Semenischev, and seminar participants at the University of Muenster for valuable comments and suggestions.

1 Introduction

Since the financial crisis, volatility markets experienced an enormous upswing in trading volume. In this paper, we study two possibilities to trade volatility. Investors can either use a portfolio which combines S&P500 (SPX) and volatility (VIX) options or a position in VIX futures. Both strategies have the same payoff and thus have by theory the same price. However, we find that the law of one price is violated. In fact, price deviations between the options and VIX futures markets can be quite large. The existence of these discrepancies raises at least two questions: Are these deviations significant or are both markets fully integrated? And if deviations are significant, are they just the result of market frictions?

In a largely model-free analysis, we find that on average the mispricing between the options and VIX futures markets can be explained by market frictions. Here, the main drivers are liquidity risk and limiting restrictions in the options availability. Liquidity risk explains almost all of the price dispersions before and during the financial crisis. Afterwards, the explanatory power is dampened for contracts with a short time to maturity. However, for longer maturities results remain unchanged. For short-term contracts we find a lead-lag structure when price deviations between both markets are severe. Implied volatility smiles show that if options are more costly than VIX futures, priced volatility risks in options adjust. If prices of VIX futures are above their option implied arbitrage bounds, future prices adjust. Therefore, the information flow between the two markets (VIX futures and SPX/VIX options) depends on which product implies higher volatility risk and, as a result, the lead-lag structure is not just one-sided.

Our analysis relies on the fact that squared VIX futures can be replicated in a model-free manner as the difference of a SPX options and a VIX options portfolio. The first captures expected forward volatility risk, the second captures the expected variation of the VIX futures, and thus vol-of-vol risk. We show that the

SPX portfolio makes up a large part of the short-term futures price, whereas the VIX options portfolio is important for long-term futures. The composition of the replication portfolio emerges directly from no-arbitrage pricing arguments. Limits of arbitrage could stem from limited strike ranges in the involved option portfolios. Therefore, we study the impact of this form of market incompleteness on the performance of the replication, which is similar to the analysis of Jiang and Tian (2007) for the VIX. We generate stock and volatility option prices in a controlled model environment for different strike grids. The studied integrated market model features jumps in the volatility and price dynamics. Our results indicate that without using interpolation techniques, market incompleteness yields small relative pricing errors in the range of 2% to 5%. If we use interpolation methods the error becomes a magnitude smaller, which is in general less than 1%. Especially for empirical strike ranges we find the errors to be small. Our analysis shows that market incompleteness is unlikely to explain all the pricing errors we observe empirically.

The portfolio of SPX options captures parts of the term structure of risk-neutral expected variance and thus relies on its measurement. We first discuss that a correct data treatment with respect to, for example, weekly and monthly options is crucial when applying the model-free formula. Second, we compare two well-known methods to build the variance term structure: the method of the CBOE (2016) and the formula of Bakshi et al. (2003). Only the latter accounts for jumps in stock prices. Our results show that if the method of the CBOE is used, the model-free VIX future replication displays significant biases. Prices are on average too low and the price deviations between both markets are significantly negatively skewed and thus highly non-normal distributed. This finding holds across all maturities. In contrast, when using the formula of Bakshi et al. (2003), pricing errors are on average zero and more symmetrically distributed. Thus, induced price deviations are substantially less skewed and therefore more normal.

Contemporaneous regressions with liquidity measures as explanatory variables show that realized price deviations between the futures and options market can be well explained by liquidity risk. Until 2010, liquidity explains about 62% to 67% of the variation in price deviations across maturities. In particular, we find that the bid-ask spread in the options market subsumes the explanatory power for liquidity risk, whereas funding liquidity has a statistically significant but small economic impact. After 2010, the explanatory power of liquidity risk for price deviations of short-term contracts with maturities between seven and 30 days declines to roughly 43%. For larger maturities liquidity risk is still the main driver with R^2s ranging from 55% to 62%. The bid-ask spread has still greatest explanatory power.

Since 2010 liquidity risk plays a lesser role for short-term VIX futures, but large price dispersions still occurred. Therefore, we study these obvious discrepancies in more detail. We define model-free upper and lower bounds for the VIX futures price in dependence of bid and ask prices of its replicating portfolio. We concentrate our analysis on implied volatility smiles of the SPX and VIX market for the days around the date where the future's price is either below or above its lower or upper bound. The smiles show that option markets do not move on average one day before the occurrence of the price deviations. A price deviation is a signal for market movements on the subsequent days. If options imply higher volatility risks compared to VIX futures, implied volatility levels of both option markets (SPX and VIX) adjust and decrease the following day. On the contrary, if futures are more costly than options, the implied volatilities do not react and, consequently, futures become less expensive. Our results document that if price deviations are quite large, then the product which implies higher volatility risks follows the cheaper one. Therefore, we find that the information flow is between VIX futures and SPX/VIX options and not between the VIX and the SPX derivatives markets.

Our research is related to the literature on model-free pricing of VIX futures, which is pioneered by Carr and Wu (2006) and Dupire (2006). They show that futures can be replicated by the expected variation in stock prices over the next 30 days in some future point in time and the expected variance of the futures itself. In addition, the CBOE suggests pricing of VIX futures using the VIX term structure and estimating the futures variance using historic data.[1] However, all the authors remain silent about the performance of the replication. We show that using the CBOE's VIX term structure leads to systematic biases in the replicated VIX futures price. Further, we relate to the literature on robust estimation of expected volatility from option prices. Similar to Jiang and Tian (2005, 2007) for the VIX, we conduct a sensitivity analysis for model-free futures pricing.

The paper which is closest to us is Park (2015). Park researches lead-lag structures between the SPX and VIX derivatives markets. Thereby, he uses the model-free valuation method as well, but rewrites it and looks on *price dislocations* between both markets. He then calibrates a model to his time-series and thereby merges the manifold information from different maturities to a single output. As a result, all his findings stem from a model-based analysis. We deviate from his procedure in several dimensions. First, our analysis is model-free and grounds on a replication strategy only. Therefore we rely solely on no-arbitrage conditions and do not use a parametrized model in our analysis. Second, we look at different maturity buckets separately and analyze the replication quality for each of them. Third, we compare the VIX futures and the options markets. Fourth, we are the first to research the sensitivity to input data of the model-free approach.

The paper is structured as follows. The next chapter describes the model-free valuation method, implementation pitfalls, and the impact of limited strike ranges. Chapter three discusses and reasons empirical performances. Here, we also show market reaction to large price deviations between markets. Chapter four concludes.

[1] See http://cfe.cboe.com/education/vixprimer/features.aspx.

2 Model-Free VIX Futures Evaluation

2.1 Theory

VIX futures can be evaluated in a model-free way as the difference of a SPX and a VIX options portfolio. This evaluation method is known since the introduction of the VIX. Its theory is described by e.g. Carr and Wu (2006) and Dupire (2006). Since the model-free formula for futures solely relies on prices of S&P500 and VIX derivatives, it provides a measure of the integrity of futures and options markets when it comes to trading volatility. The formula is essentially a hedge for futures. Consequently, a violation of the pricing formula could directly lead to arbitrage opportunities between both markets by exploiting the hedge relation. According to no-arbitrage pricing theory, the replication formula should work perfectly. However, little is known about its real-world performance. If pricing errors of significant magnitude exist, the question arises if these errors can be explained by market frictions or if it gives indeed arbitrage opportunities. The latter would imply that the options and futures market are not perfectly integrated. The following theorem describes the pricing of VIX futures using portfolios of S&P500 and VIX derivatives.

Theorem 1 (Model-Free Valuation of VIX futures) *The squared VIX futures price* $\left(F_t^T\right)^2$ *is the difference of the expect forward variance* $\mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right]$ *and a convexity correction*

$$\left(F_t^T\right)^2 = \mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] - [Convexity\ Correction]_t, \tag{1}$$

$$\mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] = \frac{1}{30D}\left((T+30D)\left(VIX_t^{T+30D}\right)^2 - T\left(VIX_t^T\right)^2\right), \tag{2}$$

$$[Convexity\ Correction]_t = 2e^{rT}\left(\int_{F_t^T}^{\infty} \mathcal{C}_t^{VIX}(T,K)dK + \int_0^{F_t^T} \mathcal{P}_t^{VIX}(T,K)dK\right), \tag{3}$$

where $\{\mathcal{C}_t^{VIX}(T,K)\}_K$ *and* $\{\mathcal{P}_t^{VIX}(T,K)\}_K$ *are prices of puts and calls for VIX options with strike* K*.* VIX_t^T *is, depending on our later analysis, the volatility index* *for maturity T calculated using the method from the CBOE (2016) or Bakshi et al. (2003), respectively. For a proof of the theorem see Appendix A.1.*

The model-free formula not only depends on the expectation about future VIX^2 values, but on a *convexity correction* as well. This correction is given by a portfolio of VIX options, which leads to a direct exposure to volatility-of-volatility risk. In our empirical section we will show that both parts of the replication portfolio are important.

In Equation (1) the right-hand side depends on the VIX futures price, because it is needed for the calculation of the convexity correction. Since our focus is on the consistency of futures and option prices, both sides of the equation should only depend on one asset class. As a work-around, we approximate the futures price via put/call-parity.

Corollary 1 (Approximative Valuation of VIX Futures) *In the setting of Theorem 1 it holds*

$$\left(F_t^T\right)^2 \approx MF_t^2(T) \equiv \mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] - [Approx.\ Conv.\ Corr.]_t, \tag{4}$$

$$[Approx.\ Conv.\ Corr.]_t = 2e^{rT}\left(\int_{\hat{F}_t^T}^{\infty} \mathcal{C}_t^{VIX}(T,K)dK + \int_0^{\hat{F}_t^T} \mathcal{P}_t^{VIX}(T,K)dK\right), \tag{5}$$

where $\hat{F}_t^T$ is the VIX futures price implied by put/call-parity of VIX options prices.[2]

Our approximation does not depend on VIX future prices as an input.[3] So we provide an independent model-free approach for pricing these products. It is worth

[2] VIX options are written on the VIX future. Thus, we can rely on put/call-parity to infer the futures price. In line with CBOE (2016), we use the option pair where the bid-prices are closest.

[3] We also run our later empirical analysis with the real futures price to compute the convexity adjustment. We find no significant changes in results, because the relative error of the VIX futures price, coming from put/call-parity almost never exceeds $\pm 1\%$.

mentioning that the proposed pricing formula in Theorem 1 has to hold even if only the right-hand side of Equation (1) is a portfolio which is directly investable. Squared futures are not traded. However, market participants can get an exposure to it, as can be seen relatively easy by Itô's lemma. The lemma implies that the dynamics of $\left(\mathrm{F}^T\right)^2$ can be replicated by taking a long-position in a VIX future and by buying a portfolio of VIX options, since it holds

$$d\left(\mathrm{F}_t^T\right)^2 = 2\mathrm{F}_t^T d\mathrm{F}_t^T + \left(d\mathrm{F}_t^T\right)^2 \approx 2\mathrm{F}_t^T \Delta\mathrm{F}_{t+1}^T + \Delta\left[\text{Convexity Correction}\right]_{t+1}, \qquad (6)$$

where $\Delta X_{t+1} = X_{t+1} - X_t$. The approximation in Equation (6) is valid, because the convexity correction equals the expected variance of the futures price. A replicable squared futures price implies that a violation of the model-free pricing formula indeed leads to an imbalance between the futures and options market.

2.2 Measures of Expected Variance

To determine the squared VIX futures price solely from option prices, we need to determine $(VIX_t^T)^2$ and $(VIX_t^{T+30D})^2$ of Equation (2). We analyze two different approaches in this paper. First, we follow the calculation method of the CBOE (2016) for the VIX, which is based on the seminal work of Demeterfi et al. (1999) on the fair value of variance, or equivalently, on the model-free implied variance of Britten-Jones and Neuberger (2000). Assuming a continuous price process, they show that the variance swap-rate approximately equals the price of a portfolio of out-of-the-money (OTM) options $O_t(K,T)$, where each option is inversely weighted by its squared strike price (K):

$$VS_t^T = \frac{2}{T}e^{rT}\int_0^\infty \frac{O_t(K,T)}{K^2}dK + \varepsilon_t^T. \qquad (7)$$

Importantly, ε_t^T is the approximation error due to discontinuous movements in the underlying price process. The CBOE's $(VIX_t^T)^2$ measure is a discretized version of

(7). It is thus a biased estimate of the variance swap rate over time T:

$$VS_t^T \approx (\mathrm{VIX}_t^T)^{2,\mathrm{CBOE}} = \frac{2e^{rT}}{T}\sum_i^N \left[\frac{\Delta K_i}{K_i^2}\, O_i^{SPX}(K_i,T)\right] - \frac{1}{T}\left(\frac{\mathcal{F}_t(T)}{K_0}-1\right)^2 . \quad (8)$$

In the first term, $O_i^{SPX}(K_i,T)$ is the i-th SPX option price, which is the mid of the bid- and ask-price. The second part of Equation (8) is a correction term which accounts for the fact that usually no option is directly traded at-the-money (ATM). The term comprises $\mathcal{F}_t(T)$, which is the S&P500 forward index level derived from SPX option prices, and K_0, which is the first strike price below $\mathcal{F}_t(T)$.[4] The CBOE's volatility index $\mathrm{VIX}^{30\mathrm{D}}$ then follows from linear interpolation, using two maturities with $T^- \leq 30D$ and $T^+ \geq$30D

$$\mathrm{VIX}_t^{30D} = \sqrt{\left\{T^-\left(VIX_t^{T^-}\right)^2 \frac{T^+-30D}{T^+-T^-} + T^+\left(VIX_t^{T^+}\right)^2 \frac{30D-T^-}{T^+-T^-}\right\}\frac{365}{30}} . \quad (9)$$

Although VIX futures are written on the $\mathrm{VIX}^{30\mathrm{D}}$, we address the question whether its jump-induced error ε_t^T also affects the model-free VIX futures pricing. Therefore, we construct $(VIX_t^T)^2$ using the model-free measure of implied variance of Bakshi et al. (2003), which comprises the possibilities of jumps in the S&P500. Based on the quadratic variation, the no-arbitrage relation and the assumption that the price process grows at the risk-free rate under the risk-neutral measure, Bakshi et al. (2003) demonstrate that the implied measure of variance can be estimated by

$$(VIX_t^T)^{2,BKM} = \frac{e^{rT}}{T}\sum_i^N \left[2\left(1-\ln\left(\frac{K_i}{S_t}\right)\right)\frac{\Delta K}{K_i^2}\, O_i^{SPX}(K_i,T) - \mu_t(T)^2\right], \quad (10)$$

where the function $\mu_t(T)$ is given in Appendix A.2. Since $\mu_t(T)^2$ is normally quite small, the sensitivity towards return jumps (compared to the CBOE's measure in Equation (8)) manifests in the additional weighting term $1-\ln(K_i/S_t)$, which depends on the options' moneyness. Since only OTM options enter the calculation, this

[4] As described below in Section 3.1, we follow Carr and Wu (2007) and interpolate the available strike range, making the correction term dispensable.

shifts weight from OTM calls to OTM puts, whereby the latter are more sensitive to negative jumps in the S&P500.

Clearly, both measures $(\mathrm{VIX}_t^T)^{2,\mathrm{CBOE}}$ and $(\mathrm{VIX}_t^T)^{2,\mathrm{BKM}}$ are subject to estimation errors. In the sense of Jiang and Tian (2005, 2007), we will analyze next how market incompleteness in the form of limited and discontinuous strike ranges affects the model-free valuation of VIX futures. This will give a first intuition of *unavoidable* pricing errors. Finally, the usage of mid-prices instead of actual trade-prices leaves us with a time-varying range of uncertainty about market participants' notion of expected variance. We will address this issue further in the empirical analysis.

2.3 Theoretical Impact of Limited Strike Ranges

Markets are not complete. For options, only a discrete set of strikes within some minimal and maximal range is traded. This induces an error when building the replicating portfolios from Equation (1), because it is not possible to trade all options that are theoretically needed. The model-free valuation method depends two-fold on the available strikes in both markets. First, options are not traded in a continuum and, thus, a discretization error enters the above integrals. Second, the limited strike range leads to a truncation error. Consequently, for the analysis of the performance of the model-free pricing approach, it is indispensable to know about the effects of market incompleteness with respect to the strike range and to the number of available strikes. Jiang and Tian (2005, 2007) show that for the VIX itself the error can be reduced substantially by using interpolation over strikes in the implied volatility space and extrapolation by using the implied volatility of the minimal and maximal available strikes. However, they do not consider the implications for errors in the implied VIX future pricing. From a practical point of view, it is further of special importance to look at pricing errors if no interpolation method is employed and only the available set of strikes is used.

In the following, we look on the impact of the availability of strikes in a model with simulated data and consider different degrees of market incompleteness. We thereby abstract from transaction costs, bid-ask spreads, liquidity constraints and other market frictions. This study gives a first intuition for the pricing errors which can be expected in our later empirical study of the model-free formula. Furthermore, it gives a range of how much of the error can be explained by market incompleteness. To generate the data we use the jump-diffusion model (SVJJ) with simultaneous jumps in the stock and its volatility introduced by Duffie et al. (2000). The model dynamics of the log stock price s_t and variance V_t are given as follows

$$ds_t = \left(r - \bar{\mu}\lambda_J - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^s + Z_t^s dN_t, \tag{11}$$

$$dV_t = \kappa_V\left(\bar{V} - V_t\right)dt + \sigma_V\sqrt{V_t}dW_t^V + Z_t^V dN_t, \tag{12}$$

$$dW_t^s dW_t^V = \rho,\ Z_t^V \sim \mathrm{Exp}\left(\mu_V\right),\ Z_t^s \mid Z_t^V \sim \mathcal{N}\left(\mu_J + \rho_J Z_t^V, \sigma_J^2\right), \tag{13}$$

where N_t is a Poisson process with constant intensity $\lambda_C > 0$, $\bar{\mu} = \frac{\mu_J + \frac{1}{2}\sigma_J^2}{1-\rho_J\mu_V} - 1$ and the remaining coefficients are chosen such that $r, \lambda_J, \sigma_V, \kappa_V, \bar{V}, \mu_V, \sigma_J > 0$ and $\rho, \rho_J, \mu_J < 0$. Since the model features stochastic volatility and jumps, it provides a valid market framework to study the model-free pricing formula for VIX futures.[5] For our analysis we use a reasonable calibration which yields a mean model-implied VIX of 25.01% and where jumps occur once every six months with an average jump size of 8% for the variance and -12% for stocks.[6] In this model setup, SPX options and VIX options and futures are known in semi-closed form. The SPX options prices are calculated as in Duffie et al. (2000), the VIX derivatives are computed as in Branger et al. (2015). From the latter, we take the analytical VIX futures price and compare

[5]Several authors find evidence for jumps in stock returns and volatility. See e.g. Eraker (2004), Todorov and Tauchen (2011), Cremers et al. (2015) amongst others.

[6]We choose the calibration to be similar to the reported parameters in Duffie et al. (2000). These parameters are $r = 0, \rho = -0.8, \rho_J = -0.4, \sigma_J = 10^{-4}, \sigma_V = 0.14, \kappa_V = 3$. The initial variance is set to its long-run mean.

it with the model-free futures prices, which follow from Equations (4) and (10), for different strike grids.

We start by analyzing the truncation error, i.e. by the limited strike range. The upper panel of Table 1 (truncation method) gives the relative valuation of the replication portfolio if we do not extrapolate in the moneyness dimension. In the lower panel (extrapolation method), we give the results if the volatility surface is extrapolated constantly as in Jiang and Tian (2007). We report errors for different moneyness ranges from ATM $\pm$ 10% to ATM $\pm$ 50% as well as for the average empirical ranges.[7] The upper panel shows that the truncation error decreases rapidly for wider ranges. For a range of ATM $\pm$ 10% the relative error for the short maturity is -15.53%, which drops with increasing moneyness range to -2.18% for ATM $\pm$ 20%. In the case of empirically observed ranges the relative error for one month futures is -0.97%. Results are similar for long-term VIX futures, but the decrease in pricing error is weaker. The error is at first enormous with -51.02% (ATM $\pm$ 10%) and decreases to -4.08% (ATM $\pm$ 40%). For the empirical moneyness ranges we find an error of -3.24%. Thus, we expect the (truncation) error to be larger for long-term futures in the empirical analysis. The lower panel shows that the errors can be reduced substantially if we extrapolate the implied volatility space constantly by using the volatility of the highest and lowest strike, respectively. For the short and long maturity the relative pricing error is below 1%-point for all moneyness ranges that are wider than ATM $\pm$ 10%. All in all, the analysis shows that a truncated strike range has little impact on the model-free VIX future valuation, especially when using empirical ranges. Our results suggest that the truncation error is roughly -1% to -3% for short-term to long-term VIX futures. As in Jiang and Tian (2005, 2007), the small relative pricing errors almost vanish if a simple extrapolation method is used.

[7] For the SPX market, we find for our sample average empirical ranges to be 0.69 to 1.15 and 0.56 to 1.30 for short and long-term options, respectively. For VIX options we find empirical ranges of 0.60 to 1.80 and 0.64 to 1.80 for short and long-term options, respectively.

Next, we concentrate on the discretization error and research the impact of a limited number of strikes. In Table 2 we give the discretization error for different combinations of SPX (rows) and VIX (columns) grids for a moneyness range of ATM $\pm$ 50%. In Panel A of Table 2 we refrain from interpolation. For one month and twelve month to maturity the valuation error is -5.37 and -2.23%, respectively, if SPX options and VIX options are available at their empirical moneyness steps (1.2 $\times S_0$ and 0.08 $\times F_0^T$). If we widen the grid for VIX options, the columns in Panel A show that the error worsens. Hereby, the number of available volatility derivatives is more important for long-term than for short-term futures.[8] Subtracting now the valuation errors for the empirical grids in Panel A from the errors for the truncation error in Table 1 for the moneyness range of ATM $\pm$ 50%, gives us an estimate for the discretization error. We find for short-term futures an error of -5.92% $(= -5.37\% - 0.55\%)$ and for long-term futures -2.28% $(= -2.23\% - 0.05\%)$. The decreasing error in the futures' maturity is plausible since the volatility smile becomes flatter for long-term options, which lowers the discretization error. If we interpolate across strikes, Panel B shows that the available number of SPX and VIX options hardly matters. For various grids, the resulting error is not different from the corresponding truncation error in Table 1, if we are in the medium state of the economy. To test whether results depend significantly on the shape of the volatility term structure, we also look at high and low volatile regimes in Panel C of Table 2. We find the relative pricing error in each volatility regime to be quite small and slightly increasing in the level of volatility. For each volatility regime the absolute pricing error is below 1%.

To sum up, we find that the discretization error can amount to roughly -6% for short-term VIX futures and roughly -2.25% for long-term futures. If we interpolate

[8] This is in line with our later findings in the empirical analysis that the convexity correction matters more for long-term futures.

in the strike dimension, the error almost vanishes. Therefore, we will also interpolate across strikes in the next section, where we look at the model-free VIX futures pricing performance empirically. Since we refrain from extrapolation in the moneyness (and maturity) space, we expect relative pricing errors of at least $1-3\%$ in an absolute sense, due to the truncation error.

3 Empirical Analysis

3.1 Data

The model-free VIX futures valuation requires a set of S&P500 options for the VIX^2 term structure and VIX options for the convexity correction term.[9] Both components need to be measured with care, since the VIX and its futures price are quoted on an annual basis. Thus, little errors in the estimation of VIX^2 can lead to enormous pricing errors. In line with that it is important to note that SPX and VIX derivatives settle at different dates. While SPX options settle on Fridays, VIX derivatives settle on Wednesdays. To correct for this two-day difference, some interpolation in the maturity dimension is required. As a consequence, we choose to only interpolate the VIX^2 term structure linearly as in Equation (9) and refrain from any inter- or extrapolation of the VIX options and futures in the maturity dimension.[10] Furthermore, we only consider those days where the maturity is straddled by the SPX options, i.e. we discard days where an extrapolation of the maturity range would be necessary.

[9]The VIX term structure is available at the CBOE via `http://www.cboe.com/data/volatilityindexes/volatilityindexes.aspx.` The time series starts on November 24th, 2010.

[10]In undocumented results, we find that a minimum of interpolation in the maturity dimension is inevitable to obtain reasonable results. For example, interpolating both VIX and SPX volatility surfaces in order to generate constant maturity contracts deteriorates pricing performances sharply.

Due to annualization, it is further highly important to account for recording and settlement times. Our S&P500 and VIX options data come from option metrics, which is recorded at the end of regular trading hours. This is 3.15 p.m. for *standard* S&P500 options and VIX options. Since 2014, however, the CBOE further includes weekly options (SPXW) into the calculation of the VIX.[11] The trading of SPXW options closes 15 minutes earlier and also the settlement is different compared to the *standard* options. The first are deemed to expire at the close of trading, the latter are deemed to expire at the opening sales price on the settlement day (8.30 a.m.). Thus, the times to maturity used for calculating VIX^2 are

$$\begin{aligned} T_{\text{standard}} &= (\text{Date}_T - \text{Date}_t - 1)/365 + \frac{(8.75 + 8.5)}{24 \times 365} \,, \\ T_{\text{weekly}} &= (\text{Date}_T - \text{Date}_t)/365 \,. \end{aligned}$$

Note that the difference in maturity is not negligible. For example, not accounting for the different settlement times of weekly options for a time to maturity of 10 days would imply an underestimation of 2.9% of VIX^2, due to biased annualization.[12]

To obtain reliable option quotes for the calculation of VIX^2 and the convexity adjustment, we follow the CBOE's white paper for filtering rules. We delete zero bids and delete all data points after two subsequent zero bids. As the CBOE does, we obtain the forward price F_t in Equation (8) (which determines OTM options) in a model-free manner by using put/call-parity and the two closest call and put prices. The call and put prices with the first strike value below the forward price

---

[11] We follow the CBOE in using only *standard* SPX options and include SPXW options, starting in 2014. Before, weekly options were not liquid enough. Thus we delete all options with the root 'JXA', 'JXB', 'JXC', 'JXD' , 'JXE'. Up to 2010, they referred to weekly options. Also, we delete options with the root 'QSE', 'QSZ', 'QZQ', 'SAQ', 'SKQ', 'SLQ', 'SQG', 'SQP', 'SZQ', 'SZU'. These are non-standard LEAPS options, which settle at the last trading day of the quarter. For further details see Andersen et al. (2011).

[12] To see this, calculate $(9/365 + 17.25/(24 \cdot 365))^{-1} \,/\, (10/365)^{-1} - 1 = 0.0289$.

are averaged.[13] In contrast to the CBOE, we further require that at least 5 option quotes (SPX and VIX options) are available and we also delete option quotes with maturities less than 7 days in order to avoid microstructure effects. Finally, we interpolate the implied volatility curve for each maturity to avoid the discretization bias, discussed by Jiang and Tian (2005) and Section 2.3. Thus, we only keep options where the implied volatility could be calculated. We interpolate linearly within the available strike range, using a fine grid of strikes with $\Delta K = K_{i+1} - K_i = 1$ for SPX options and $\Delta K = 0.1$ for VIX options.[14] For the SPX options we use Equations (8) or (10) to measure VIX_t^T. The convexity correction CC_t^T in Equation (5) is then calculated via

$$CC_t^T = 2e^{rT} \sum_{i=1}^{n} O_i^{\text{VIX}}(K_i, T)\Delta K_i, \tag{14}$$

where $O_i^{\text{VIX}}(K_i, T)$ are the OTM VIX option prices.

Our VIX futures quotes are directly obtained from the CBOE's website. We only keep traded futures, i.e. futures with non-zero volume, and futures where the convexity correction CC_t^T could be calculated (due to enough VIX option quotes). The two-day difference in settlement days and the filtering rules on (SPX) options further restrict us on VIX futures with maturity of more than 8 days. We further only consider futures with maturities of less than six months.

VIX futures started trading in 2004, VIX options followed in February 2006. The latter, however, were at first highly illiquid, which sets our available time-span (due to the restrictions of at least 5 traded VIX options). All in all, the filters leave us with 2,168 days of data in the time-span from September 1, 2006, to end of August, 2015. In total we have 11,859 futures prices. Figure 1 plots the daily trading volume of these futures prices, averaged over each month. It shows that the trading

[13] For further details see CBOE (2016).

[14] We find no significant changes when we change the interpolation method.

volume remained rather low for all maturities until the mid of 2009. Since then, market participants' interest in short-term futures rose rapidly and since January 2010 trading of VIX futures increased across all maturities. For this reason, we look at the overall sample and also subdivide it at January 4, 2010. We refer to the two subsample as *pre- and in-crisis* and *post crisis* sample. Finally, note that the drop in short-term futures volume at the end of our sample period reflects the CBOE's introduction of weekly VIX futures on July 23, 2015. Since weekly VIX options followed later on October 8, 2015, we can only study *standard* VIX futures prices.

3.2 Performance of the Model-Free Formula

In this section we test the performance of the model-free VIX futures pricing formula with real world data. We use Equation (4) to price futures, using only information from SPX and VIX options. Before we start with the empirical pricing performance of the model-free VIX futures valuation, we discuss the importance of the expected VIX^2 and the convexity adjustment. Their descriptives are given for the shortest maturity and the whole sample in Panel A of Table 3.[15] For brevity, the table only reports the shortest maturity, whereas Figure 2 illustrates the findings for the shortest and longest maturity bucket. On average the expected VIX^2 is 6.08%, which is roughly half a percentage point higher than the squared VIX futures price. Interestingly, the importance of the convexity correction is increasing in maturity. While for short-term futures it is rather unimportant with a level of 5.96% of the short-term futures price, its relative contribution increases in maturity up to roughly 20% for the longest maturity (T_6). The top panel of Figure 2 shows further that the absolute convexity adjustment is rather stable at low levels if markets are calm. But in times of market stress, the convexity adjustments become quite large. The

[15] We compute the VIX^2 term structure for this analysis by the method of Bakshi et al. (2003). Results are similar if we apply the methodology of the CBOE.

lower panel shows that in the course of the financial crisis of 2008 and at the peaks of the European sovereign debt crisis in 2010 and 2011, it made up for 17% in relative terms of the short-term VIX futures price and for more than 35% of the long-term VIX futures price. Compared to the expected VIX^2, Table 3 shows further that the second to fourth moments of the convexity adjustment are much higher. Altogether, we find that the convexity adjustment becomes more important the longer the maturity of the VIX future and the more volatile the market gets. This is model-free evidence that stochastic vol-of-vol risk is priced in VIX futures and that it is especially important for longer maturities.

We now compare the methods of the CBOE and Bakshi et al. (2003) to construct the VIX term structure with respect to pricing errors. To do so, we analyze the whole term structure of VIX futures and report pricing errors sorted by maturity buckets. Now, Table 4 reports the relative and absolute pricing errors over the whole sample for the two different methods to calculate the VIX term structure. The table states errors and summary statistics for six different maturity buckets starting from short-dated futures with maturities from seven to 30 days, up to long dated futures with time to maturity of more than 150 days. Panel A of Table 4 documents the pricing error if the VIX term structure is calculated using the approach of the CBOE. We focus on short-term futures with maturities no more than 30 days, on the mid-term bucket with maturities between 60 and 90 days and on long-term futures with time to maturity of more than 150 days.

We find the relative pricing errors, which we define as the model-free future over the futures price minus one, to be negative for all maturities. For short-term futures the average (median) relative error is -2.79% (-2.82%), for mid-term futures it is -7.20% (-4.39%), and for long-term futures it is -7.22% (-5.08%). Further, we find the quantiles for these errors to be unbalanced. In absolute values, the lower 5% quantile is much higher than the upper 5% quantile. The upper quantiles are

3.29%, 6.57% and 7.82% for the short-, mid-, and long-term futures, whereas the lower 5% quantiles are -8.79%, -29.97% and -29.24%. Thus, the quantiles are highly skewed and document that model-free futures prices, which are computed by CBOE method, are systematically too low. The pattern of the absolute errors are the same.

Figure 3 visualizes the pricing errors for the CBOE approach. Generally, the magnitude of errors is smallest for the shortest maturity and increases in market volatility. Especially in the financial crisis the pricing errors became quite large. During the crisis model-free futures prices for maturities of more than 30 days even became negative. Looking at the VIX^2 term structure on a *de-annualized* basis, we find the reason for negative prices in an expected variation of the stock market, which is decreasing in maturity. This implication from the data is theoretically impossible and shows an inconsistent pricing of the underlying SPX options.[16] In general, the errors in VIX futures pricing seem to be biased downward, meaning that the model-free price is systematically too low. Overall, the figure as well as Panel A of Table 4 document that the pricing errors can become quite negative and have huge standard deviations. Comparing these errors with our results on limited option availability, we conclude that the latter is not the only source of the errors, because they are too large.

Panel B of Table 4 shows the pricing errors when using the method of Bakshi et al. (2003) to calculate the VIX term structure. In this case, we find average and median pricing errors across all maturity buckets closer to zero than with the VIX term structure coming from the CBOE method. The average (median) relative errors are 0.80% (-0.90%) for the short-term bucket, -2.89% (0.24%) for the mid-term bucket, and 0.11% (2.40%) for the long-term bucket. Further, we find the unbalance between the upper and lower 5% quantile to be less pronounced than for the CBOE method. For the upper (lower) 5% quantiles we find values of 7.36%

[16]By theory it hold that $\int_0^{t+s} (d\ln S_u)^2\, du \geq \int_0^{t} (d\ln S_u)^2\, du$ for all $s \geq 0$.

(-6.66%), 16.03% (-31.66%) and 21.97% (-30.75%) for short-, mid-, and long-term futures. The fit and absolute errors are shown in Figure 4. It is evident from the plot that the pricing errors are larger for longer maturities. As before, pricing errors are more severe during times of higher market volatility. In contrast to the CBOE method, the pricing errors are overall more symmetrically distributed. However, from the standard deviation and quantiles it is clear that the relative errors are still too large to be fully explained by market incompleteness in terms of option availability. The ranges of errors well exceed the values for the truncation error, suggested by our theoretical analysis in Section 2.3.

Our overall results indicate that the method for the calculation of the VIX term structure is crucial for model-free futures pricing. If the method of the CBOE is used, futures pricing errors are large in an absolute and relative sense and resulting futures prices are systematically too low. This is evident from average and median errors as well as from our quantile analysis. In comparison, if the Bakshi et al. (2003) approach is used, relative and absolute errors are smaller and less biased. This is especially true for maturities below 60 days and thus the most liquid contracts. For all maturities, we find that the differences of the two approaches are highly significant with t-statistics exceeding 9. Thus the methodology for estimating the term structure of expected variance highly matters for the model-free VIX futures pricing.

3.3 Liquidity and Pricing Errors

In this section we aim to explain the emerging pricing errors. As a result of the previous section, we choose to analyze the pricing errors using the method of Bakshi et al. (2003) to calculate the VIX term structure. The reason is that with this method the pricing errors are less biased and on average closer to zero as expected from a valid replication strategy. Further, we are interested in differences of the preand in-crisis compared to the post-crisis period, because after the financial crisis trading of volatility derivatives has significantly increased. After the crisis, pricing errors might be less dependent on liquidity measures in comparison to the pre- and in-crisis.

To analyze the driving forces behind the pricing errors, we conduct contemporaneous regressions of absolute pricing errors for our two subsamples, which cover the period from September 1, 2006 to December 31, 2009 and from January 1, 2010 to August 31, 2015, respectively. For the regressors, we choose liquidity measures and control for different states of the economy by volatility risk measures. A widely used measure of liquidity is the bid-ask spread. In the following we define a weighted spread for SPX and VIX options. The weighting gives the spreads for the two parts of the model-free Formula (4).

As multiple options enter the model-free futures price, we define two weighted spreads of the involved bid- and ask prices of the SPX and VIX options. The weight of each option is determined by its contribution to the VIX term structure or the convexity correction, respectively. The two spreads are then given by

$$\mathrm{Spread}_{t,T}^{\mathrm{SPX}} \equiv \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2\right]^{\mathrm{ask}} - \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2\right]^{\mathrm{bid}} \tag{15}$$

$$\mathrm{Spread}_{t,T}^{\mathrm{VIX}} \equiv \mathrm{CC}_{t,T}^{\mathrm{ask}} - \mathrm{CC}_{t,T}^{\mathrm{bid}} \,, \tag{16}$$

which can also be interpreted as the spread of the forward expected VIX^2 and the convexity correction. Further, we include the unweighted average volume of SPX and VIX options. We include all these variables to proxy for hedging costs and liquidity constraints. In addition, we include the TED-Spread as a proxy for funding liquidity.[17] To control for the state of the economy we also include the VIX from the CBOE as well as the VVIX. The former captures overall volatility risk and the latter

[17] See e.g. Gupta and Subrahmanyam (2000) and Campbell and Taksler (2003).

volatility-of-volatility risk, which is especially relevant for VIX option prices.[18]

All in all, the regression takes the form

$$\epsilon_t^{MF^2,i} = \alpha + \beta X_t + \gamma \,[\text{Interact. Terms}]_t + \eta_t \tag{17}$$

where $\epsilon_t^{MF^2,i}$ is the absolute pricing error of the squared VIX futures price, given by $MF_t^2(T_i) - \left(F_t^{T_i}\right)^2$. We normalize the regressors X_t by their subsample's standard deviation to ensure comparability of the betas and further include their interaction terms to account for their correlation.[19] Table 3 reports descriptives of our variables. In the full and both sub-samples, the absolute spreads of options are larger than the spreads of VIX options, but in relative terms they are quite the same.[20] Their higher moments are similar, with exception of the standard deviation which is slightly higher for SPX options. The aggregate daily volume of SPX options is on average twice as high as for VIX options, with 0.47 million trades per day compared to 0.29 million. Compared to all other regressors, the volume variables are non-persistent with an autocorrelation of 0.50 and 0.69 for SPX and VIX options respectively. In contrast, the TED-spread is highly persistent (AR(1)=0.99) with an average value of 0.51. Its high kurtosis reflects the liquidity dry up within the crisis of 2008.

Table 6 and Table 7 report results of the regressions with and without interaction terms for both subsamples and for the three maturity buckets of 7 to 30 days (short-term), 60 to 90 days (mid-term), and more than 150 days (long-term). First, we conduct three separate restricted regressions for option-implied spreads, the market conditions and for the volume measures and the TED-spread. Compar-

[18]See e.g. Park (2016).

[19]Table 5 reports on the correlations of the variables contained in X.

[20]It is not surprising that the absolute spreads in the SPX market are larger than in the VIX market, since the underlying price is larger as well. If the absolute spreads are normalized, e.g. by the average $\mathbb{E}\left[\left(\text{VIX}_{T_1}^{30\text{D}}\right)^2\right]$ and the convexity correction, the relative spreads are 15.13% and 15.15% for the SPX and VIX market, respectively.

ing subsequently the adjusted R^2s of these restricted regressions with unrestricted regressions gives us the economic importance of each variable. We start with Table 6, which presents the results without interaction terms. Afterwards, we discuss the impact of the latter using Table 7.

Panel A of Table 6 reports the results for the pre- and in-crisis period. For short-term contracts the first restricted regression reveals that the spread in SPX options increases the pricing error and is highly significant. The weighted bid-ask spreads explain more than half of the variation in pricing errors and yield an impressive R^2 of 56.06%. However, the spread of the VIX market is insignificant. This finding is in line with its minor impact on the short-term model-free futures price as discussed in the previous section. The beta of the SPX spread is positive, which makes sense considering that the model-free futures price is the difference of a portfolio of SPX and VIX options. So the more expensive SPX options the more costly is the replicated (model-free) futures price. The results for maturities T_3 and T_6 in Panel A show that the spread of the SPX market remains significant and also the bid-ask spread of VIX options becomes highly significant. Again, this in line with the intuition that the convexity correction term gets more relevant for larger maturities. Its negative beta stems from the fact that VIX options enter the model-free replication as a short position. For the mid-term and long-term contracts we find adjusted R^2s of 66.77% and 61.80%, respectively.

Turning now to the restricted regressions on volatility measures, we find that the VIX is only relevant for short-term futures. For the first maturity bucket the VIX is highly significant with an R^2 of 28.90%, whereas the VVIX is insignificant. For longer maturities results are mixed with respect to statistical significance, but the adjusted R^2s drop to almost zero.[21] Thus, we find that the VIX is only relevant for contracts with the shortest time to maturity.

[21] We find the same if we use the VIX with the corresponding maturity T_i for each bucket.

In the third restricted regressions, we concentrate on the trading volume of options and funding liquidity measured by the TED-Spread. The table documents insignificant results for trading volumes across all maturities. The TED-Spread is highly significant for short maturities with an adjusted R^2 of 13.68%, but unimportant for longer maturities.

Unrestricted regressions which include all the former variables have only small marginal explanatory power of roughly $2-4\%$ compared to the restricted regressions which only include the spread measures. This result is even more strongly pronounced in Table 7 which includes interaction terms. If we control for these terms, the significance of Spread$^{\text{SPX}}$ vanishes for maturity bucket T_1 due to the high correlation between the variables. Nevertheless, judging from the changes in R^2s, the spreads in the option markets are most important since they subsume most of the relevant informations contained in the other variables. Overall, Panel A shows, that in the pre- and in-crisis period price deviations between the VIX futures market and the option markets can almost be fully explained by our bid-ask measures, thus by the options' liquidity.

Finally, Panel B of Table 6 documents results for the post-crisis period. For this period, we find similar results with respect to sign and significance of the betas as for the first period. So the overall pattern remains unchanged. However, we identify three major differences. First, the scaled beta estimates are lower in absolute terms. The reason is that pricing errors are smaller after the financial crisis. Second, the spread in VIX options becomes more relevant for longer maturities than one month, since its significance does not vanish anymore in unrestricted regressions. Panel B of Table 7 documents that this holds true even after controlling for interaction terms. Third and most importantly, we find that the maximal explanatory power of our variables decreases for short-term contracts, whereas it remains rather similar for the other ones. If the bid-ask spread measures are included, we find that for VIX futures with maturities between seven and 30 days the explanatory power drops by 20%, i.e. the R^2s of the full regressions in Table 6 and 7 drop from roughly 60% to 40%. In contrast, we do not find such large drops in R^2-values for mid- and long-term contracts in the post-crisis period. The pattern holds also if we include the interaction terms. All in all, the spread in the options markets explains most of the pricing errors across all maturities and both samples. Still, a rather big part of the variation of errors (40%-60%) cannot be explained after the crisis. Thus, we cannot rationalize these price deviations, which suggest that market frictions between the options and futures markets exist. Since the price deviations can be least explained for short-term contracts, we analyze next how market participants react to most inconsistent prices in volatility contracts.

3.4 Market Reactions to Inconsistent Pricing

In the previous section we studied the drivers of price deviations of volatility products in the VIX futures and options markets. Our results indicate that the deviation varies a lot and can take negative as well as positive signs. Thus, there are times were VIX futures are more expensive relative to options and vice versa. This section aims to uncover the market reaction to such price differences. We focus on most obvious inconsistencies between the VIX futures and the replication portfolio. Thereto, we define upper and lower bounds for VIX futures which depend on the prices of the option portfolios only. Since we find in the previous section that price dispersions of short-maturity contracts can be least explained by liquidity risks in the period after the financial crisis, we study options and futures for this period with maturities between 7 and 30 days in more detail.

The main idea is to define the bounds for futures prices in dependence of options' bid- and ask-prices, because we do not know the exact trading price for the options on a particular day. We only know the upper and lower ranges of these prices and use them to define our bounds. Thus, if the replication strategy holds, the VIX futures price has to lie within these bounds.[22] In each point in time the bounds have to hold, otherwise futures are too expensive/cheap relative to options and profits can easily be made by taking a position in futures and the hedge portfolios. To define the model-free bounds we rely on Equation (4). The replication portfolio would be cheapest (most expensive) if the necessary SPX options with time to maturity $T+30D$ trade at their bid-prices (ask-prices) and VIX options as well as SPX option with time to maturity in T trade at their ask-prices (bid-prices). Therefore, we define the model-free lower bound for VIX futures as the cheapest price of the replication portfolio. The price is the difference between the bid-price for the expected squared VIX and the ask-price of the convexity correction, given by

$$\mathcal{L}_t^T \equiv \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)\right]^{\mathrm{bid}} - \mathrm{CC}_{t,T}^{\mathrm{ask}} \, . \tag{18}$$

Similar, we define a model-free upper bound for VIX futures as the ask-price of the expected squared VIX. We do not subtract the (positive) convexity correction and thus get a stronger upper bound

$$\mathcal{U}_t^T \equiv \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)\right]^{\mathrm{ask}} \, . \tag{19}$$

Empirically, the arbitrage bounds are violated and we observe times where $F_t^T < \mathcal{L}_t^T$ or $\mathcal{U}_t^T < F_t^T$. Note that by looking at violations of these bounds, we concentrate on times when it may be difficult to explain price deviations with the bid-ask spread in SPX and VIX options as done in our previous regression analysis.

Figure 5 shows the relative pricing error for short-term futures with respect to the bounds given that the futures price is either too low or too high. We find that the lower bound is violated more often and in larger magnitude than the upper

[22] Unfortunately, our data only covers mid-prices of the VIX futures. This is, however, no limitation to our results, because the relative bid-ask spread of VIX futures almost never exceeds 1% (see Park (2015)).

bound. Thus, in our sample options were too expensive more often relative to VIX futures, rather than the other way around. For the whole sample we find that for 10.19% of the days the future is below its lower bound with an average relative error of 9.75%, whereas the upper bound is violated only on 3.21% of the days and with an average relative error of -4.57%. Further, we find that the times where the lower bound $\mathcal{U}_t^T$ is violated are evenly spread across the sample, whereas futures exceeded their upper bound mostly in the financial crisis.

A violation of the model-free bounds hints to dispersions between the options and VIX futures market. We are thus interested in how the markets react to a violation of the bounds and how the price deviations are resolved. Panels B and C of Table 3 document market descriptives in times when the upper and lower bound is violated. Panel B shows that if the futures price is above its upper bound, $\mathcal{U}^{30\text{D}}$, the convexity correction and CBOE's VIX index are lower than on average. Even the spread in SPX options is only half of the average spread. So VIX futures are more expensive than options on relatively calm days. This is in contrast to our findings for the subsample when the VIX future is below its lower bound. In these times we find that futures, $\mathbb{E}\left[\left(\text{VIX}_{T_1}^{30\text{D}}\right)^2\right]$, the convexity correction and the bid-ask-spreads as well as CBOE's VIX are well above their average values. Hence, when the upper bound is violated, the market environment tends to be more rough than usual.

Figure 6 shows implied volatility smiles for SPX and VIX short-term options from a kernel regression on transformed moneyness $m = \log(K/F_0) \,/\, \left(\sqrt{T}\sigma_{\text{ATM}}\right)$.[23] We report smiles on the day of the violation of the upper and lower bound, as well as one and two days before and after the bound's violation. For comparison, we include the average smile in the plots.

[23]The definition for the maturity-adjusted moneyness is used by other authors as well, for example Andersen et al. (2016). Note, that in Section 3.1 we pointed at the necessity to use two SPX option portfolios, due to the two-day difference in settlement days. For illustrative purposes, we only show smiles for the shorter end of SPX options' maturity.

If the futures price is below its lower bound, VIX futures are cheaper relative to options, or put differently, volatility risks implied by VIX futures are lower. In this case we find a strong reaction in both volatility smiles (SPX and VIX) after the day of this price deviation. The day before and the exact day of the violation option smiles remain almost unchanged. The plots in the third row of Figure 6 show that the level of both volatility smiles decrease the day afterwards. Thus, the information implied by the price deviation triggers market movements around its date of occurrence.

As pointed out earlier, SPX options account on average for more than 94% of the value of the short-term model-free replication and they positively influence its price. Therefore, volatility risk embedded in SPX option prices decreases to mitigate the relative overpricing. This means that the level of aggregate volatility risk, implied by SPX options, decreases. For VIX options it could have been expected that they are cheaper in the case of an overpricing, since they have a negative impact on the replication portfolio. However, they are more expensive at and before the day of the mispricing and their implied volatilities decrease afterwards. This means that not only the level of volatility risk decreases, but also its uncertainty (volatility-of-volatility risk). This leads to overall lower volatility risks implied by the option markets. Thus, our results indicate that if VIX futures are relatively cheaper than options, the expectation about future volatility risks in the option markets follow.

In the case when futures are overpriced relative to options, i.e. the upper bound is violated, we find almost no reaction in the option markets. Figure 6 documents that starting from the day just before the dispersion to two days after it, the level of the smile of SPX options decreases slightly, whereas the level of implied volatility of the VIX options remains mostly unchanged. This shows that the upper bound is violated on days when the expected volatility traded in the SPX market decreases. The drop in SPX smiles and the lack of an increasing pattern after the mispricing implies that prices of VIX futures adjust when they are relatively overpriced compared to options.

Overall we find that the volatility product that is less expensive sets expected volatility risks. The volatility embedded in the cheaper products adjusts. Either the prices of futures adjust or the implied volatility of options. Our results indicate a lead-lag structure between the market for VIX futures and the markets for SPX and VIX options. Consequently, we observe a lead-lag structure between markets segmented by their product, not by their underlying.

4 Conclusion

This paper studies the interdependencies between the VIX futures market and the SPX and VIX options markets using model-free methods. The main interest of our analysis is the model-free VIX futures replication by a long position in a portfolio of SPX options and a short position in a portfolio of VIX options. We conduct an extensive sensitivity analysis and find that limited option availability only leads to relative small pricing errors. An application to real world data shows that the pricing errors are too large to be explained by market incompleteness (in terms of limited strikes) alone. Thereby, we analyze the impact of the construction of the VIX term structure, which is necessary to build the SPX replication portfolio. We compare the approach of the CBOE (2016) with the formula of Bakshi et al. (2003) and show that the latter results in less skewed, and thus more reasonable pricing errors. The construction with the CBOE method leads to a systematic underpricing and strongly negatively skewed pricing errors. For the pre- and in-crises period (before 2010), we reason the price deviations across all VIX futures' maturities with liquidity risk. After the crisis we cannot explain large parts in the deviations for short-term contracts with maturities smaller than 30 days. For contracts with larger time to maturity we still identify liquidity risk as the main driver. Thus, we further analyze market reactions to price deviations in short-term contracts and find that either futures prices or options prices in both markets (S&P500 and VIX) adjust. We conduct this analysis by studying implied volatility smiles and therefore remain model-free. If VIX futures imply lower volatility risks, SPX and VIX option prices adjust. On the other hand, if options imply lower volatility risks, VIX futures adjust. We thus uncover a lead-lag structure between the VIX futures and the options market in times when price dispersions are largest.

A Appendix

A.1 Proof of Theorem 1

Carr and Madan (2001) show that for a twice differential function $f : \mathbb{R} \mapsto \mathbb{R}$ holds

$$f(X) = f(\bar{X}) + f'(\bar{X})(X - \bar{X}) + \int_{\bar{X}}^{\infty} f''(K)(X - K)^+ dK + \int_0^{\bar{X}} f''(K)(K - X)^+ dK, \tag{20}$$

for $X, \bar{X} \in \mathbb{R}$. If we set $f(X) = X^2$, $X = \mathrm{VIX}_T$ and $\bar{X} = \mathbb{E}_t[\mathrm{VIX}_T] = \mathrm{F}_t^T$, it follows

$$\begin{aligned}\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2 &= \left(\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\right)^2 + 2\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\left(\mathrm{VIX}_T^{30\mathrm{D}} - \mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\right) \\ &\quad + 2\int_{\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]}^{\infty} (\mathrm{VIX}_T^{30\mathrm{D}} - K)^+ dK + \int_0^{\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]} (K - \mathrm{VIX}_T^{30\mathrm{D}})^+ dK.\end{aligned}$$

Now taking expectation on both sides and rearranging yields

$$\left(\mathrm{F}_t^T\right)^2 = \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2\right] - 2e^{rT}\left(\int_{\mathrm{F}_t^T}^{\infty} \mathcal{C}_t^{\mathrm{VIX}}(T,K)dK + \int_0^{\mathrm{F}_t^T} \mathcal{P}_t^{\mathrm{VIX}}(T,K)dK\right). \tag{21}$$

A.2 Option-Implied Implied Variance of Bakshi et al. (2003)

Let $\mathcal{C}_t$ denote OTM call option prices C_t(K,T) and $\mathcal{P}_t$ denote OTM put option prices P_t(K,T), both with with strike prices K and maturity T. Bakshi et al. (2003) demonstrate that the annualized risk-neutral expected variance over period T ($\mathbb{E}_t[V_{t,T}]$) can be estimated from a portfolio of $\mathcal{C}_t$ and $\mathcal{P}_t$, given by

$$\mathbb{E}_t[V_{t,T}] = \frac{e^{rT}}{T}\left[\underbrace{\int_{K>S_t} \frac{2(1 - \ln(K/S_t))}{K^2}\,\mathcal{C}_t\,dK + \int_{K<S_t} \frac{2(1 + \ln(S_t/K))}{K^2}\,\mathcal{P}_t\,dK}_{H_1} - \mu_t(T)^2\right]$$

where $\mu_t(T)$ is

$$\mu_t(T) = 1 - \frac{1}{e^{rT}} - \frac{H_1}{2} - \frac{H_2}{6} - \frac{H_3}{24},$$

with the three hypothetical securities H_i, which pay quadratic, cubic and quartic payoffs, respectively. H_i are again portfolios of options. H_1 is already defined above and H_2 and H_3 are given by

$$
\begin{aligned}
H_2 &= \int_{K>S_t} \frac{6\ln(K/S_t) - 3(\ln(K/S_t))^2}{K^2}\, \mathcal{C}_t\, dK \ \ldots \\
&\ldots\ - \int_{K<S_t} \frac{6\ln(S_t/K) + 3(\ln(S_t/K))^2}{K^2}\, \mathcal{P}_t\, dK \\
H_3 &= \int_{K>S_t} \frac{12(\ln(K/S_t))^2 - 4(\ln(K/S_t))^3}{K^2}\, \mathcal{C}_t\, dK \ \ldots \\
&+ \int_{K<S_t} \frac{12(\ln(S_t/K))^2 + 4(\ln(S_t/K))^3}{K^2}\, \mathcal{P}_t\, dK.
\end{aligned}
$$

Our measure $(VIX_t^T)^{2,BKM}$ is the discretized version of $\mathbb{E}_t[V_{t,T}]$. To circumvent errors due to the discretization, we interpolate the volatility smile in the strike dimension at each day and for each maturity with one-point increments.

References

Andersen, T. G., O. Bondarenko, and M. T. Gonzalez-Perez (2011): "Coherent model-free implied volatility: A corridor fix for high-frequency VIX," *CREATES Research Papers*, 49, 2011–49.

Andersen, T. G., N. Fusari, and V. Todorov (2016): "Short-Term Market Risks Implied by Weekly Options," *Working Paper*.

Bakshi, G., N. Kapadia, and D. Madan (2003): "Stock Return Characteristics, Skew Laws, and Differential Pricing of Individual Equity Options," *Review of Financial Studies*, 16, 101–143.

Branger, N., A. Kraftschik, and C. Völkert (2015): "The Fine Structure of Variance: Consistent Pricing of VIX Derivatives," Working Paper.

Britten-Jones, M. and A. Neuberger (2000): "Option Prices, Implied Price Processes, and Stochastic Volatility," *Journal of Finance*, 55, 839–866.

Campbell, J. Y. and G. B. Taksler (2003): "Equity volatility and corporate bond yields," *The Journal of Finance*, 58, 2321–2350.

Carr, P. and D. Madan (2001): "Optimal positioning in derivative securities," *Quantitative Finance*, 1, 19–37.

Carr, P. and L. Wu (2006): "A Tale of Two Indices," *Journal of Derivatives*, 13, 13–29.

——— (2007): "Stochastic Skew for Currency Options," *Journal of Financial Economics*, 86, 213–247.

CBOE (2016): "The CBOE Volatility Index – VIX," White Paper.

Cremers, M., M. Halling, and D. Weinbaum (2015): "Aggregate Jump and Volatility Risk in the Cross-Section of Stock Returns," *Journal of Finance*, 70, 577–614.

Demeterfi, K., E. Derman, M. Kamal, and J. Zou (1999): "A Guide to Volatility and Variance Swaps," *Journal of Derivatives*, 4, 9–32.

Duffie, D., J. Pan, and K. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343–1376.

Dupire, B. (2006): "Model free results on volatility derivatives," *Working Paper*.

Eraker, B. (2004): "Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices," *Journal of Finance*, 59, 1367–1403.

Gupta, A. and M. Subrahmanyam (2000): “An empirical examination of the convexity bias in the pricing of interest rate swaps,” *Journal of Financial Economics*, 55, 239–279.

Jiang, G. and Y. Tian (2005): “The Model-Free Implied Volatility and Its Information Content,” *Review of Financial Studies*, 18, 1305–1342.

——— (2007): “Extracting Model-Free Volatility from Option Prices: An Examination of the VIX Index,” *Journal of Derivatives*, 14, 1–26.

Park, Y.-H. (2015): “Price Dislocation and Price Discovery in the S&P 500 Options and VIX Derivatives Markets,” *Working Paper*.

——— (2016): “The effects of asymmetric volatility and jumps on the pricing of VIX derivatives,” *Journal of Econometrics*, 192 (1), 313–328.

Todorov, V. and G. Tauchen (2011): “Volatility Jumps,” *Journal of Business and Economic Statistics*, 29, 356–371.

Min. Moneyness Max. Moneyness	0.9 1.1	0.8 1.2	0.7 1.3	0.6 1.4	0.5 1.5	Emp. Ranges
	Truncation Method					
1 Month to Mat.	-15.53	-2.18	0.15	0.53	0.55	-0.97
12 Months to Mat.	-51.02	-28.47	-13.55	-4.08	0.05	-3.24
	Extrapolation Method					
1 Month to Mat.	-15.53	-0.31	0.38	0.54	0.55	0.12
12 Months to Mat.	0.24	-0.12	0.29	0.92	0.03	0.88

Table 1: Relative Errors of Model-Free VIX Futures in the SVJJ Model

The table shows the relative pricing errors in percentage points of the model-free formula for VIX futures for different available strike ranges in the SVJJ model of Duffie et al. (2000). Available minimal and maximal strikes are quoted in moneyness terms (K/F). For the column *Emp. Ranges* we use average empirical strike ranges for the SPX and VIX market. For the SPX (VIX) market, average ranges in our sample are 0.69 to 1.15 (0.60 to 1.80) and 0.56 to 1.30 (0.64 to 1.80) for short- and long-term options, respectively. In each case we interpolate strikes in the implied volatility space using linear interpolation with 1000 nodes. The initial stock price is $S_0 = 100$ and the initial variance V_0 is at its long-term mean.

Panel A	Without Interpolation					
Δ Strike	Emp. F_0^T	0.025 F_0^T	0.05 F_0^T	0.075 F_0^T	0.1 F_0^T	0.25 F_0^T
	1 Month to Maturity					
Emp. S_0	-5.37	-5.37	-5.39	-5.41	-5.45	-5.81
	12 Month to Maturity					
Emp. S_0	-2.23	-2.23	-2.25	-2.27	-2.31	-2.76

Panel B	With Interpolation					
Δ Strike	Emp. F_0^T	0.025 F_0^T	0.05 F_0^T	0.075 F_0^T	0.1 F_0^T	0.25 F_0^T
	1 Month to Maturity					
Emp. S_0	0.55	0.55	0.55	0.55	0.55	0.55
0.050 S_0	0.55	0.55	0.55	0.55	0.55	0.55
0.075 S_0	0.53	0.53	0.53	0.53	0.53	0.54
0.100 S_0	0.46	0.46	0.46	0.46	0.46	0.46
	12 Month to Maturity					
Emp. S_0	0.03	0.03	0.03	0.03	0.03	0.04
0.050 S_0	0.03	0.03	0.03	0.03	0.03	0.03
0.075 S_0	0.04	0.04	0.04	0.04	0.04	0.04
0.100 S_0	0.04	0.04	0.04	0.04	0.04	0.05

Panel C	Different Volatility Regimes		
Time to Maturity	Volatility Regime High	Medium	Low
1 Month	0.71	0.55	0.14
12 Month	-0.09	0.03	0.25

Table 2: Relative Errors of Model-Free VIX Futures in the SVJJ Model

The table shows the relative pricing errors in percentage points of the model-free formula for VIX futures for different available strikes in the SVJJ model of Duffie et al. (2000). Available strikes are quoted in moneyness terms. For the interpolation in Panels B and C, we interpolate in the volatility space. For both markets we choose the available strike ranges as ATM $\pm$ 50%[*Underlying*]. We use $V_0 = 0.053$ (its long-term mean) for the sensitivity analyzes with respect to the strike grid. For the analysis over different volatility regimes we use empirical strike grids with step-size $0.012 \times S_0$ for stock options and $0.008 \times F_0^T$ for VIX options. In the *High* (low) regime, we define the initial variance V_0 as 0.17 (0.01). In the *Medium* regime the initial variance is at its long-term mean.

	$\left(F^{T_1}\right)^2$	$\mathbb{E}\left[\left(\mathrm{VIX}^{30\mathrm{D}}_{T_1}\right)^2\right]$	Conv. Corr.	$\epsilon^{MF^2,1}$	$\mathrm{Spread}^{\mathrm{SPX}}_{\mathrm{T}_1}$	$\mathrm{Spread}^{\mathrm{VIX}}_{\mathrm{T}_1}$	VIX	VVIX	$\mathrm{VOL}^{\mathrm{SPX}}$	$\mathrm{VOL}^{\mathrm{VIX}}$	TED-Sprd
Panel A	Moments: Full Sample 09/01/2006 to 08/31/2015										
Mean	5.54	6.08	0.33	0.21	0.92	0.05	21.26	86.49	0.47	0.29	0.51
Std	5.58	6.60	0.45	1.08	1.35	0.07	10.2	13.11	0.18	0.25	0.56
Skew	3.01	3.48	4.37	4.68	5.54	6.08	2.21	0.97	1.47	2.08	2.82
Kurt	13.66	18.27	29.16	49.09	45.70	59.51	9.21	4.81	7.41	10.62	13.16
AC1	0.98	0.97	0.92	0.65	0.79	0.81	0.98	0.93	0.50	0.69	0.99
Panel B	Moments: Subsample $\left(F^{T_1}\right)^2 >$ Upper Bound after 01/04/2010										
Mean	4.93	4.41	0.28	-0.80	0.45	0.05	19.53	81.20	0.44	0.17	0.65
Std	5.79	4.84	0.52	1.64	0.67	0.07	12.35	14.49	0.16	0.20	0.63
Skew	2.41	2.37	3.87	-3.66	3.70	3.63	2.08	0.98	0.51	1.48	1.60
Kurt	8.08	8.17	19.99	17.99	18.26	17.68	7.25	3.23	2.70	4.08	4.41
AC1	0.84	0.84	0.71	0.58	0.46	0.72	0.86	0.67	0.49	0.81	0.82
Panel C	Moments: Subsample $\left(F^{T_1}\right)^2 <$ Lower Bound after 01/04/2010										
Mean	9.29	11.49	0.61	1.62	1.79	0.09	27.05	90.59	0.51	0.27	0.77
Std	9.99	12.65	0.81	2.30	2.74	0.14	15.88	16.04	0.18	0.27	0.79
Skew	1.68	1.79	2.20	2.51	2.94	3.57	1.38	1.19	1.08	2.32	1.66
Kurt	4.66	5.19	7.57	9.35	12.79	19.46	4.00	5.77	4.67	11.45	5.33
AC1	0.97	0.97	0.86	0.82	0.73	0.68	0.95	0.79	0.24	0.64	0.96

Table 3: Descriptives of Pricing-Errors and Liquidity-Measures

The table shows descriptives of pricing errors and liquidity measures for the full sample from 09/01/2006 to 08/31/2015 and for two subsamples. $\left(F^{T_1}\right)^2$ is the squared VIX futures price for the first maturity bucket, $\epsilon_t^{MF^2,i} = MF_t^2(T_i) - \left(F_t^{T_i}\right)^2$ is the pricing error for the model-free VIX futures from Equations (4) and (10). $\mathrm{Spread}^{\bullet}_{T_1}$ are the weighted bid-ask spreads for the SPX and VIX options market, calculated by Equations (15) and (16). VIX and VVIX are the volatility and volatility-of-volatility indices from CBOE. $\mathrm{VOL}^{\bullet}$ are the daily aggregate trading volumes in millions for VIX and SPX options. TED-Spread is the difference between the 3-month LIBOR and the T-Bill rate.

Maturity	Mean	Median	Std	$q^{0.95}$	$q^{0.05}$
Panel A	CBOE Method for VIX TS				
	Relative Error				
7D $<$ T$_1$ $\leq$ 30D	-2.79	-2.82	3.74	3.29	-8.79
30D $<$ T$_2$ $\leq$ 60D	-4.10	-3.83	5.89	3.76	-11.94
60D $<$ T$_3$ $\leq$ 90D	-7.20	-4.39	14.62	6.57	-29.97
90D $<$ T$_4$ $\leq$ 120D	-0.85	-0.58	13.02	15.94	-19.14
120D $<$ T$_5$ $\leq$ 150D	-4.30	-2.69	11.81	8.08	-20.70
150D $<$ T$_6$	-7.22	-5.08	14.41	7.82	-29.24
	Absolute Error				
7D $<$ T$_1$ $\leq$ 30D	-0.0064	-0.0056	0.0096	0.0065	-0.0227
30D $<$ T$_2$ $\leq$ 60D	-0.0101	-0.0080	0.0163	0.0074	-0.0333
60D $<$ T$_3$ $\leq$ 90D	-0.0192	-0.0094	0.0461	0.0150	-0.0841
90D $<$ T$_4$ $\leq$ 120D	-0.0025	-0.0012	0.0393	0.0425	-0.0525
120D $<$ T$_5$ $\leq$ 150D	-0.0114	-0.0059	0.0363	0.0214	-0.0514
150D $<$ T$_6$	-0.0187	-0.0115	0.0428	0.0193	-0.0772
Panel B	BKM Method for VIX TS				
	Relative Error				
7D $<$ T$_1$ $\leq$ 30D	0.08	-0.09	4.32	7.36	-6.66
30D $<$ T$_2$ $\leq$ 60D	-0.30	0.00	7.16	9.98	-10.25
60D $<$ T$_3$ $\leq$ 90D	-2.89	0.24	17.83	16.03	-31.66
90D $<$ T$_4$ $\leq$ 120D	6.44	6.78	16.90	27.68	-15.44
120D $<$ T$_5$ $\leq$ 150D	3.15	4.60	15.08	20.97	-18.50
150D $<$ T$_6$	0.11	2.40	18.76	21.97	-30.75
	Absolute Error				
7D $<$ T$_1$ $\leq$ 30D	0.0013	-0.0002	0.0123	0.0198	-0.0135
30D $<$ T$_2$ $\leq$ 60D	-0.0003	0.0000	0.0202	0.0255	-0.0238
60D $<$ T$_3$ $\leq$ 90D	-0.0083	0.0005	0.0568	0.0438	-0.0884
90D $<$ T$_4$ $\leq$ 120D	0.0165	0.0139	0.0532	0.0793	-0.0413
120D $<$ T$_5$ $\leq$ 150D	0.0079	0.0100	0.0463	0.0630	-0.0397
150D $<$ T$_6$	0.0003	0.0052	0.0556	0.0643	-0.0731

Table 4: Errors of Model-Free VIX Futures

The table shows moments for the absolute and relative pricing errors (in percentage points) of the model-free formula for VIX futures for different maturity buckets. We report results for the cases when the VIX is calculated following CBOE's White Paper and Bakshi et al. (2003), respectively. The absolute error is defined as $\epsilon_i^{\text{abs}} \equiv MF_t(T_i) - F_t^{T_i}$ and the relative error as $\epsilon_i^{\text{rel}} = MF_t(T_i)/F_t^{T_i} - 1$. To calculate the moments we use daily data from 09/01/2006 till 08/31/2015.

	$\left(F^{T_1}\right)^2$	$\mathbb{E}\left[\left(\text{VIX}^{30\text{D}}_{T_1}\right)^2\right]$	Conv. Corr.	$\text{Spread}^{\text{SPX}}_{\text{T}_1}$	$\text{Spread}^{\text{VIX}}_{\text{T}_1}$	VIX	VVIX	VOL^{SPX}	VOL^{VIX}	TED-Spr.
$\left(F^{T_1}\right)^2$	1	0.99	0.84	0.83	0.78	0.97	0.30	0.21	−0.29	0.58
$\mathbb{E}_t\left[\left(\text{VIX}^{30\text{D}}_{T}\right)^2\right]$		1	0.87	0.87	0.82	0.96	0.33	0.21	−0.26	0.58
Conv. Corr.			1	0.79	0.93	0.86	0.57	0.26	−0.14	0.49
$\text{Spread}^{\text{SPX}}_{\text{T}_1}$				1	0.78	0.81	0.37	0.23	−0.17	0.49
$\text{Spread}^{\text{VIX}}_{\text{T}_1}$					1	0.78	0.51	0.24	−0.12	0.45
VIX						1	0.37	0.29	−0.29	0.63
VVIX							1	0.28	0.17	0.11
VOL^{SPX}								1	0.16	0.15
VOL^{VIX}									1	−0.39
TED-Spr.										1

Table 5: Correlations of Pricing-Errors and Liquidity-Measures: Full Sample 09/01/2006 to 08/31/2015

The table shows the correlation of pricing errors and liquidity measures for the full sample from 09/01/2006 to 08/31/2015. $\left(F^{T_1}\right)^2$ is the squared VIX futures price for the first maturity bucket. $\text{Spread}^{\bullet}_{T_1}$ are the weighted bid-ask spreads for the SPX and VIX options market, calculated by Equations (15) and (16). VIX and VVIX are the volatility and volatility-of-volatility indices from CBOE. $\text{VOL}^{\bullet}$ are the daily aggregate trading volumes in millions for VIX and SPX options. TED-Spread is the difference between the 3-month LIBOR and the T-Bill rate.

Panel A	Pre- and In-Crisis – 09/01/2006 to 12/31/2009											
Mat. bucket	$7D < T_1 \leq 30D$				$60D < T_3 \leq 90D$				$150D < T_6$			
Intercept	−0.0045***	−0.0234***	−0.0065**	0.0088	−0.0091***	0.0098	−0.0014	0.0072	−0.0079***	0.0478***	−0.0041	0.0308***
	(0.0011)	(0.01)	(0.0032)	(0.0066)	(0.0028)	(0.033)	(0.0086)	(−0.018)	(0.0027)	(0.0149)	(0.0066)	(0.0118)
$\text{Spread}^{\text{SPX}}_{\text{T}_i}$	0.0116***			0.0145***	0.0481***			0.0495***	0.0395***			0.0400***
	(0.0021)			(0.0023)	(0.0049)			(0.0049)	(0.0026)			(0.0029)
$\text{Spread}^{\text{VIX}}_{\text{T}_i}$	0.0005			0.0029	−0.0123***			−0.0028	−0.0107***			−0.0089***
	(0.0028)			(0.0024)	(0.0040)			(0.0017)	(0.0028)			(0.0032)
VIX		0.0081***		−0.0057***		−0.0060		−0.0157***		0.0085***		−0.0012
		(0.0035)		(0.0019)		(0.0086)		(0.0049)		(0.0034)		(0.0035)
VVIX		0.0016		−0.0012		−0.0010		−0.0008		−0.0107***		−0.0069***
		(0.0014)		(0.0009)		(0.0059)		(0.0030)		(0.0026)		(0.0021)
VOL^{SPX}			0.0014	0.0001			0.0022	0.0028			0.0017	0.0004
			(0.0017)	(0.0011)			(0.0031)	(0.0017)			(0.0023)	(0.0015)
VOL^{VIX}			−0.0008	−0.0000			−0.0018	−0.0001			−0.0004	−0.0001
			(0.0012)	(0.0006)			(0.0021)	(0.0011)			(0.0015)	(0.0010)
TED-Spread			0.0055***	0.0013			−0.0079	0.0038			−0.0012	0.0035**
			(0.0019)	(0.0012)			(0.0051)	(0.0026)			(0.0019)	(0.0017)
adj. R^2	0.5606	0.2890	0.1368	0.6037	0.6677	0.0083	0.0117	0.6933	0.6180	0.0293	-0.0050	0.6303
Interaction Terms	No	No	No	No	No	No	No	No	No	No	No	No
Panel B	Post-Crisis – 01/04/2010 to 08/31/2015											
Intercept	−0.0021***	−0.0062*	−0.0031***	−0.0022	−0.0026***	−0.0104***	−0.0096***	−0.0045	−0.0044***	−0.0210***	−0.0143***	−0.0195***
	(0.0004)	(0.0035)	(0.0010)	(0.0016)	(0.0007)	(0.0038)	(0.0025)	(0.0028)	(0.0012)	(0.0072)	(0.0036)	(0.0045)
$\text{Spread}^{\text{SPX}}_{\text{T}_i}$	0.0020***			0.0021***	0.0112***			0.0113***	0.0174***			0.0158***
	(0.0004)			(0.0007)	(0.0008)			(0.0009)	(0.0018)			(0.0013)
$\text{Spread}^{\text{VIX}}_{\text{T}_i}$	0.0004			0.0004	−0.0028***			−0.0023***	−0.0034***			−0.0052***
	(0.0003)			(0.0004)	(0.0006)			(0.0007)	(0.0013)			(0.0016)
VIX		0.0018***		0.0002		0.0012		−0.0009		0.0052***		0.0021*
		(0.0003)		(0.0006)		(0.0023)		(0.0012)		(0.0017)		(0.0012)
VVIX		0.0002		−0.0000		0.0010		0.0002		0.0015		0.0005
		(0.0006)		(0.0004)		(0.0010)		(0.0005)		(0.0014)		(0.0008)
VOL^{SPX}			0.0007***	−0.0004**			0.0009	0.0002			0.0014***	0.0006
			(0.0002)	(0.0002)			(0.0006)	(0.0004)			(0.0007)	(0.0006)
VOL^{VIX}			0.0002	0.0006***			0.0007	0.0010***			−0.0002	0.0014***
			(0.0004)	(0.0003)			(0.0005)	(0.0002)			(0.0006)	(0.0004)
TED-Spread			0.0006***	−0.0003			0.0023***	0.0001			0.0057***	0.0018***
			(0.0002)	(0.0002)			(0.0009)	(0.0005)			(0.0013)	(0.0008)
adj. R^2	0.3756	0.2678	0.0656	0.4070	0.6134	0.0212	0.0349	0.6182	0.5552	0.0778	0.0794	0.5785
Interaction Terms	No	No	No	No	No	No	No	No	No	No	No	No

Table 6: Dependence of Pricing Errors of the Model Free Futures on Liquidity Measures (no Interaction Terms)

The table shows betas and intercepts for the regressions of pricing errors $\epsilon_t^{MF^2,i} = MF_t^2(T_i) - \left(F_t^{T_i}\right)^2$ for the model-free VIX futures from Equation (4) on liquidity measures. We use the approach of Bakshi et al. (2003) to calculate the VIX term structure. All explanatory variables are normalized by their standard deviation. For the regressions we use daily data. *, ** and *** indicate statistical significance at the 90, 95, and 99% confidence level. Newey-West robust standard errors are stated in parentheses.

Panel A	Pre- and In-Crisis – 09/01/2006 to 12/31/2009											
Mat. bucket	$7D < T_1 \leq 30D$				$60D < T_3 \leq 90D$				$150D < T_6$			
Intercept	−0.0006	−0.0234***	−0.0065**	0.0023	−0.0084***	0.0098	−0.0014	0.0023	−0.0086***	0.0478***	−0.0041	0.0230*
	(0.0013)	(0.0100)	(0.0032)	(0.0073)	(0.0038)	(0.033)	(0.0086)	(0.0154)	(0.0030)	(0.0149)	(0.0066)	(0.0135)
$\text{Spread}^{\text{SPX}}_{\text{T}_i}$	0.0074***			0.0087	0.0436***			0.0595***	0.0410***			0.0435***
	(0.0025)			(0.0055)	(0.0116)			(0.0191)	(0.0039)			(0.0117)
$\text{Spread}^{\text{VIX}}_{\text{T}_i}$	−0.0054*			−0.0091	−0.0129***			0.0057	−0.0100***			−0.0027
	(0.0029)			(0.0086)	(0.0047)			(0.0112)	(0.0036)			(0.0055)
VIX		0.0081***		−0.0015		−0.0060		−0.0128***		0.0085***		0.0002
		(0.0035)		(0.0026)		(0.0086)		(0.0056)		(0.0034)		(0.0043)
VVIX		0.0016		−0.0001		−0.0010		−0.0006		−0.0107***		−0.0064***
		(0.0014)		(0.0011)		(0.0059)		(0.0029)		(0.0026)		(0.0023)
VOL^{SPX}			0.0014	−0.0002			0.0022	0.0022			0.0017	0.0004
			(0.0017)	(0.0009)			(0.0031)	(0.0016)			(0.0023)	(0.0016)
VOL^{VIX}			−0.0008	−0.0000			−0.0018	−0.0008			−0.0004	−0.0002
			(0.0012)	(0.0006)			(0.0021)	(0.0012)			(0.0015)	(0.0011)
TED-Spread			0.0055***	0.0009			−0.0079	0.0035			−0.0012	0.0039*
			(0.0019)	(0.0010)			(0.0051)	(0.0021)			(0.0019)	(0.0020)
adj. R^2	0.6221	0.2890	0.1368	0.6331	0.6686	0.0083	0.0117	0.7008	0.6179	0.0293	-0.0050	0.6301
Interaction Terms	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Panel B	Post-Crisis – 01/04/2010 to 08/31/2015											
Intercept	−0.0010***	−0.0062*	−0.0031***	−0.0018	−0.0007	−0.0104***	−0.0096***	0.0049*	−0.0041***	−0.0210***	−0.0143***	−0.0087*
	(0.0003)	(0.0035)	(0.0010)	(0.0019)	(0.0009)	(0.0038)	(0.0025)	(0.0028)	(0.0016)	(0.0072)	(0.0036)	(0.0053)
$\text{Spread}^{\text{SPX}}_{\text{T}_i}$	0.0010***			−0.0014*	0.0080***			0.0059***	0.0172***			0.0113***
	(0.0003)			(0.0008)	(0.0013)			(0.0018)	(0.0021)			(0.0029)
$\text{Spread}^{\text{VIX}}_{\text{T}_i}$	−0.0003			0.0005	−0.0035***			−0.0047***	−0.0037***			−0.0122***
	(0.0003)			(0.0007)	(0.0007)			(0.0015)	(0.0015)			(0.0020)
VIX		0.0018***		0.0006		0.0012		−0.0056***		0.0052***		−0.0031
		(0.0003)		(0.0006)		(0.0023)		(0.0017)		(0.0017)		(0.0020)
VVIX		0.0002		0.0001		0.0010		0.0006		0.0015		0.0013*
		(0.0006)		(0.0004)		(0.0010)		(0.0005)		(0.0014)		(0.0007)
VOL^{SPX}			0.0007***	−0.0003*			0.0009	0.0003			0.0014***	0.0007
			(0.0002)	(0.0002)			(0.0006)	(0.0003)			(0.0007)	(0.0005)
VOL^{VIX}			0.0002	0.0004*			0.0007	0.0005***			−0.0002	0.0008*
			(0.0004)	(0.0002)			(0.0005)	(0.0002)			(0.0006)	(0.0004)
TED-Spread			0.0006***	−0.0002			0.0023***	0.0004			0.0057***	0.0022***
			(0.0002)	(0.0002)			(0.0009)	(0.0005)			(0.0013)	(0.0007)
adj. R^2	0.4297	0.2678	0.0656	0.4632	0.6220	0.0212	0.0349	0.6484	0.5551	0.0778	0.0794	0.6011
Interaction Terms	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

Table 7: Dependence of Pricing Errors of the Model Free Futures on Liquidity Measures (with Interaction Terms)

The table shows betas and intercepts for the regressions of pricing errors $\epsilon_t^{MF^2,i} = MF_t^2(T_i) - \left(F_t^{T_i}\right)^2$ for the model-free VIX futures from Equation (4) on liquidity measures. We use the approach of Bakshi et al. (2003) to calculate the VIX term structure. All explanatory variables are normalized by their standard deviation. For the regressions we use daily data. *, ** and *** indicate statistical significance at the 90, 95, and 99% confidence level. Newey-West robust standard errors are stated in parentheses.

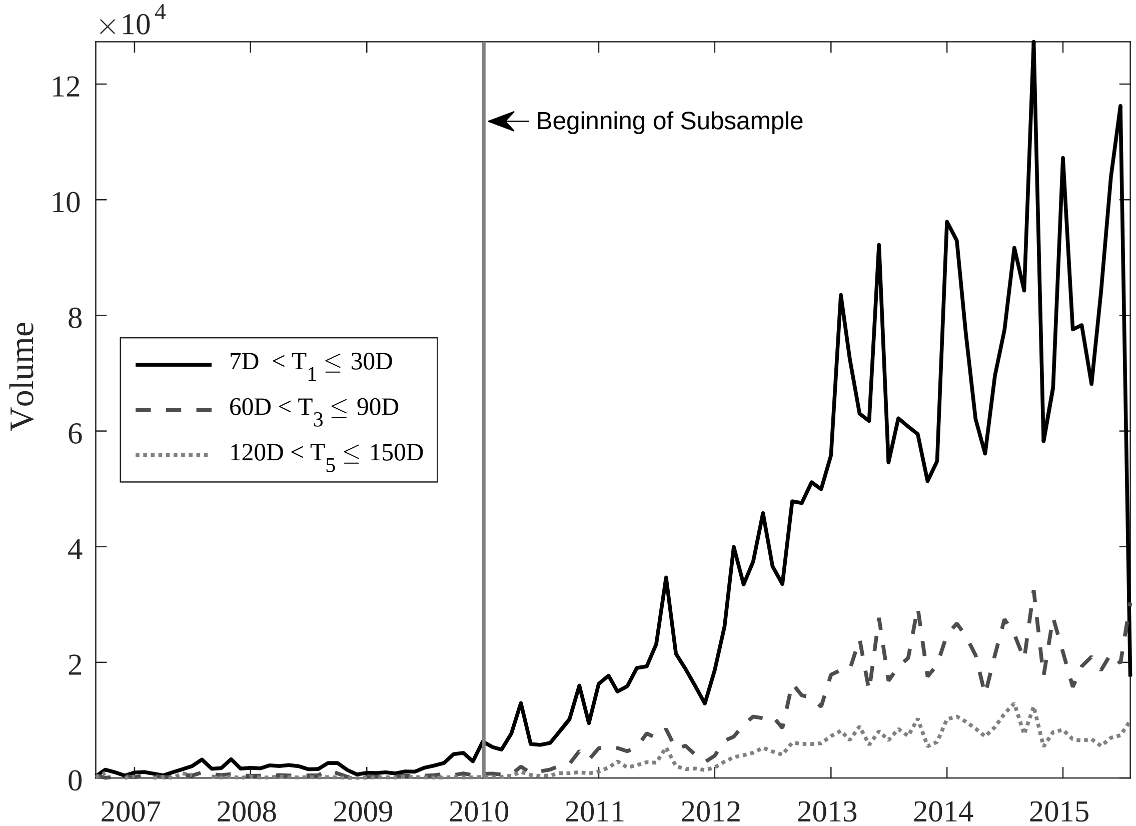

Figure 1: VIX Futures Trading Volume

The figure shows the daily volume of VIX futures, averaged over one month for different maturities. For the plot we use daily data from 09/01/2006 till 08/31/2015.

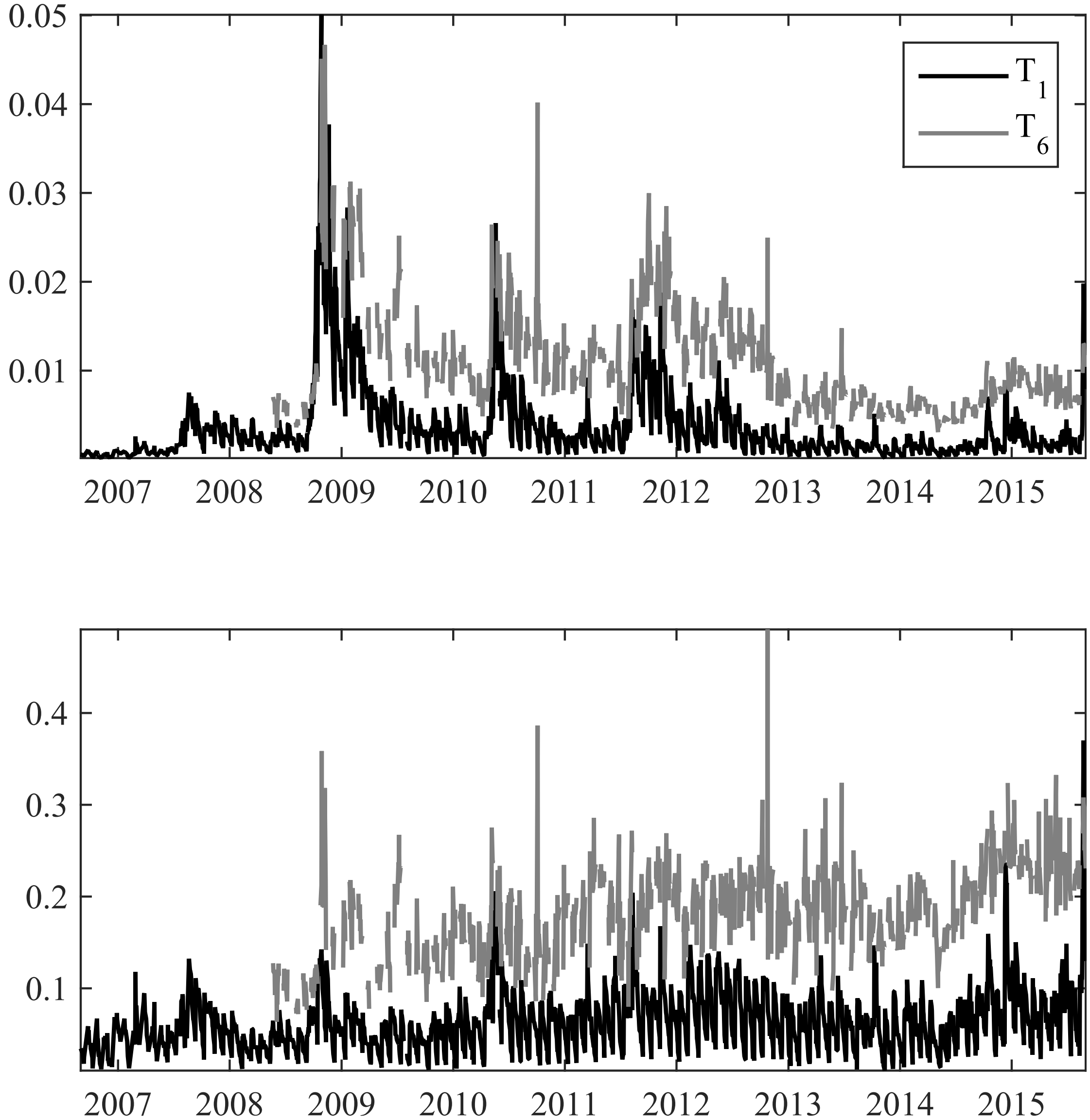

Figure 2: Convexity Correction for Maturity Buckets T_1 and T_6

The figure shows the convexity correction for the maturity buckets $7\text{D} < T_1 \leq 30\text{D}$ and $150\text{D} < T_6$. We calculate the correction using Equation (14) and the relative contribution is with respect to the *true* futures price.

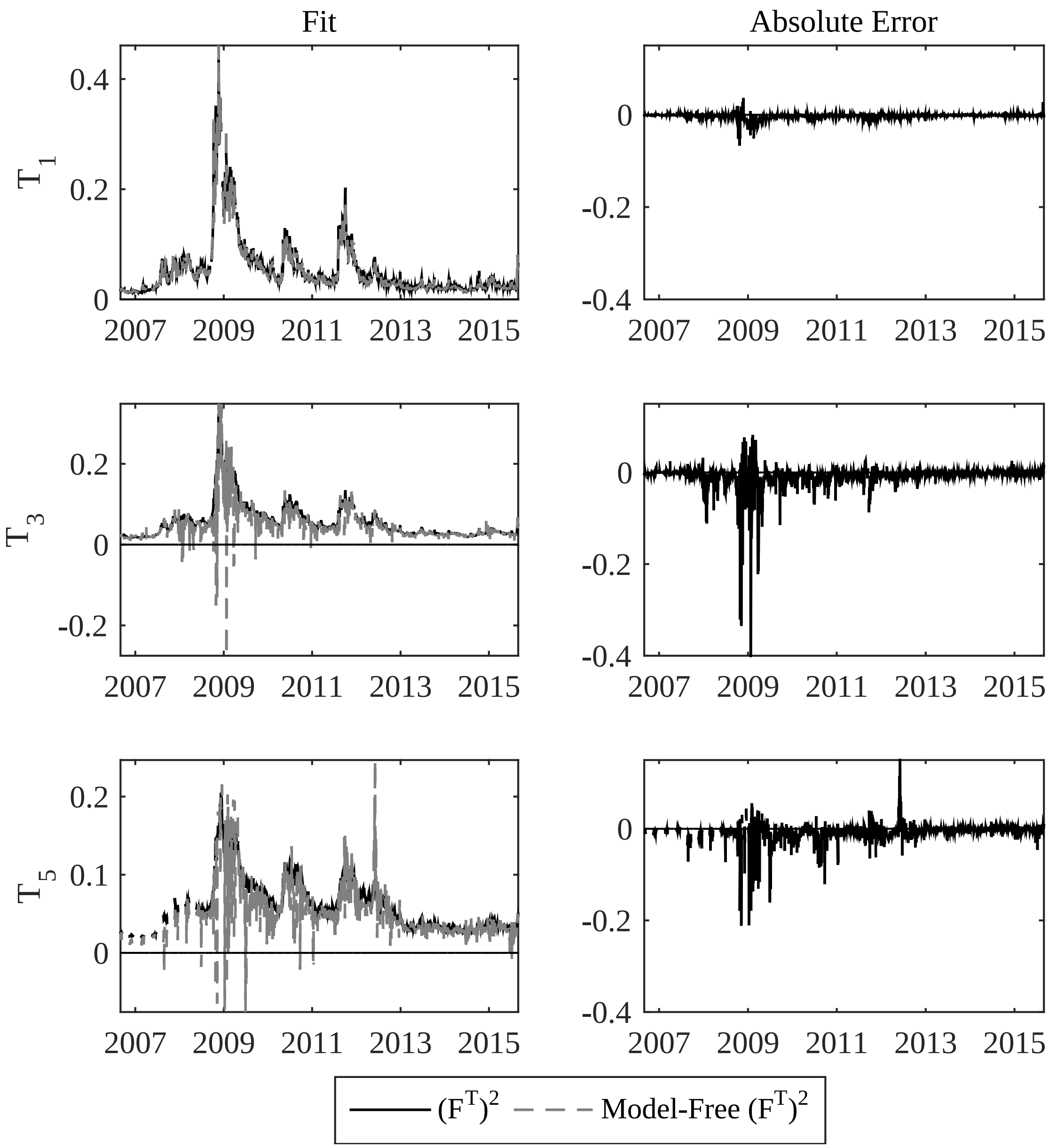

Figure 3: Model-free VIX Futures Pricing Errors: CBOE Method

The figure shows the fit for the model-free futures calculation on the left and the difference error on the right when the VIX is calculated using the approach from CBOE's White Paper. For the plots we use daily data from 09/01/2006 till 08/31/2015.

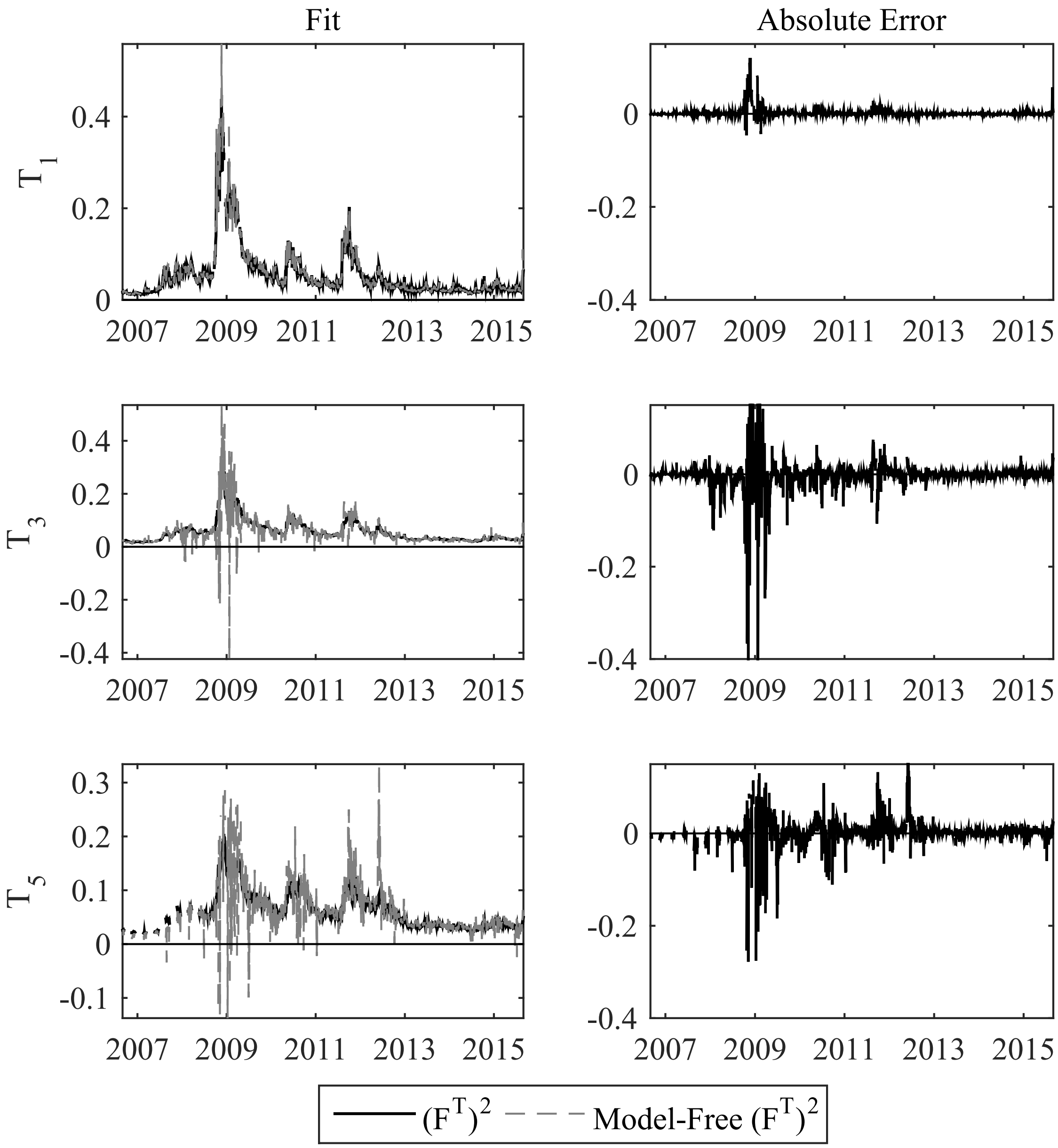

Figure 4: Model-free VIX Futures Pricing Errors: Bakshi et al. (2003) Method

The figure shows the fit for the model-free futures calculation on the left and the difference error on the right when the VIX is calculated using the approach from Bakshi et al. (2003). For the plots we use daily data from 09/01/2006 till 08/31/2015.

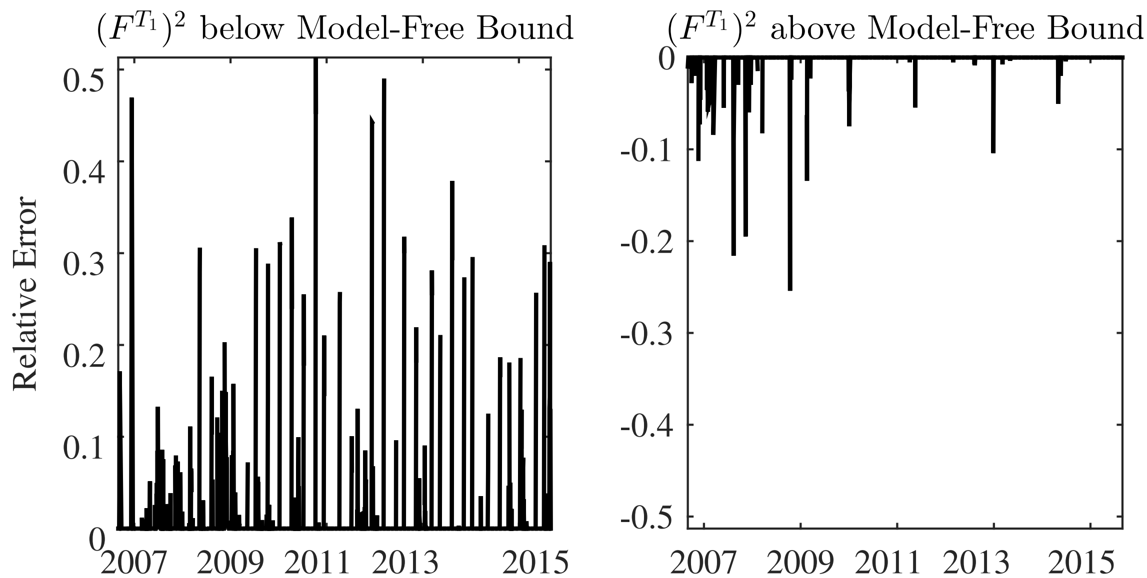

Figure 5: Relative Errors if Futures Bounds are violated

The figure shows the relative pricing error when the short-term VIX future is below its lower bound (left panel) and above its upper bound (right panel).

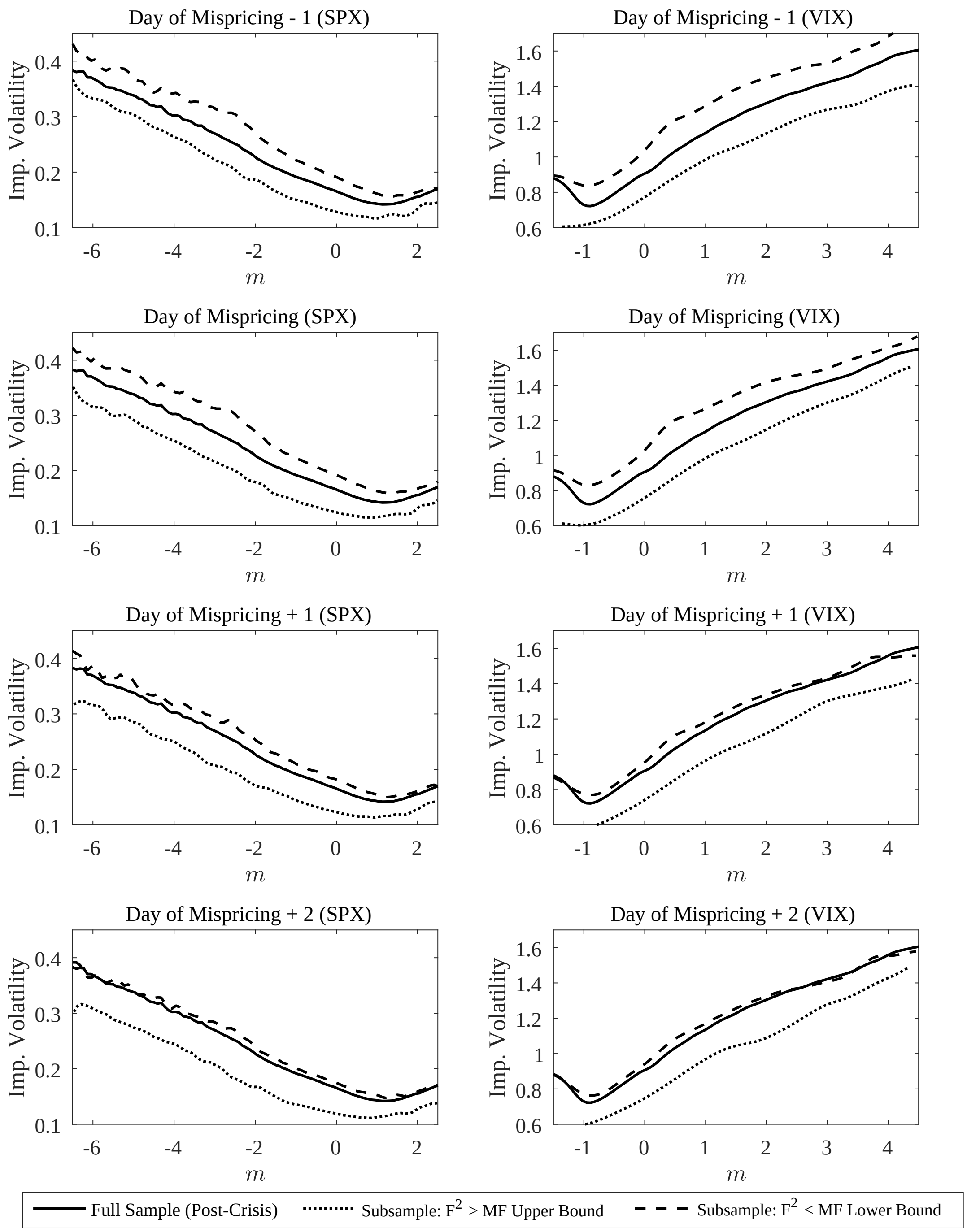

Figure 6: Market Reactions to Futures Bounds Violations

The figure shows the market reaction to price dispersion between the VIX futures and options market. The left panels shows implied volatility smiles relative to moneyness for SPX options. The right panel pictures corresponding smiles for VIX options. The smiles are obtained from kernel regressions.