The Components of Illiquidity Premium: An Unobserved Components approach

Abstract: The objective of this paper is to provide a new methodology that helps to estimate the conditional liquidity-adjusted capital asset pricing model (L-CAPM) of Acharya and Pedersen (2005). Our key novelty is that we model illiquidity via Unobserved Components (UC) models to test a *conditional* version of the L-CAPM Model. This methodology allows to take into account the main stylised facts of liquidity time series and eliminates the look-ahead bias present in previous literature (Acharya and Pedersen, 2005, Saad and Samet, 2014, Korajczyk and Sadka, 2008, Hagstromer et al., 2013). Based on a sample containing all common firms listed on the NASDAQ from 01/012006 to 12/31/2014 we obtain the following main results. In line with previous empirical studies we founding a marginal effect of liquidity risk on returns compare to the effect of the liquidity level premium. The most important liquidity risk is related to the covariance between portfolio illiquidity and market return. But, in contraction with previous founding, liquidity risk and illiquidity level are not found to be always positively correlated (i.e. we find a negative correlation between portfolio return and market illiquidity).

JEL classification : G0 ; G1 ; G12

Keywords : Liquidity risk, Conditional Liquidity-adjusted CAPM,

1 Introduction

The objective of this paper is to provide a new methodology that helps to estimate the conditional liquidity-adjusted capital asset pricing model (L-CAPM) of Acharya and Pedersen (2005). Our key novelty is that we model illiquidity via Unobserved Components (UC) models to test a *conditional* version of the L-CAPM Model. In this model, uncertainty in the illiquidity cost is what generates liquidity risk. Henceforth, estimation of this uncertainty (illiquidity innovations) appears pivotal. The current empirical literature extract these innovations via simple autoregressive models (Acharya and Pedersen, 2005, Saad and Samet, 2014, Korajczyk and Sadka, 2008, Hagstromer et al., 2013). This methodology entails several issues. Firstly and as far as we read, all the literature estimates these ARIMA type innovations via the Maximum Likelihood Estimation(MLE) method without adopting a one-step-ahead forecasting framework. This introduces a look-ahead bias since liquidity innovations obtained at the beginning of the sample use the whole sample information (the MLE is maximized with respect to the entire sample period). This undoubtedly weaken the significance of the results.

Secondly, the ARIMA methodology is based on the ideas that non-stationary series can be made stationary through the use of a specific operation. Several liquidity metrics are known to be non-stationary (for example volume-based liquidity metrics) but knowing whether the series is trend stationary or whether it is a unit root process is rarely obvious. This leads to the use of poorly theoretical-backed methods to render the data stationary, as an illustration, Acharya and Pedersen (2005) adopts a rather arbitrary method[1] to make the Amihud liquidity measurement stationary.

Thirdly, current ARIMA models applied to liquidity pricing do not take into account a number of salient features of liquidity series. For instance, seasonality known to be present in liquidity series is not seriously considered in the liquidity empirical asset pricing literature.

We propose to overcome all these limitations by modelling liquidity via Unobserved Components models (UC). UC models use a decomposition approach by explicitly modelling the components of a series where each component can be easily interpreted. This has the advantage of describing the various components of interest, making interpretation and model selection easier. Besides the Kalman Filter [2], employed to fit UC models removes the aforementioned look-ahead bias since only past data is employed to obtain state estimates. UC models have also the advantage of dealing with practical issues such as missing observations which are often observed in liquidity proxies. More importantly, our UC model specification can distinguish between permanent shocks and temporary shocks (impossible in ARIMA modelling).

The original conditional L-CAPM model assumes that betas are time-varying. However, most of the empirical literature has been focused on the unconditional version proposed by Acharya and Pedersen (2005). Few papers tested the L-CAPM in which betas are time varying (Minovic and Zivkovic, 2010; Hagstromer et al., 2013). We contribute to this literature by applying our innovative way of modelling illiquidity on a model with multiple time-varying betas.

The paper is organised as follows: the theoretical L-CAPM model is presented in section

[1] See equation 18 of [Acharya and Pedersen, 2005].
[2] Applied as a forward-looking algorithm.

2. The materiel and methodology are presented in section 3. Section 4 reports the empirical results and section 5 concludes.

2 The L-CAPM Model

The L-CAPM model introduced by Acharya and Pedersen (2005) is an extension of the classic CAPM (Sharpe; 1964; Lintner, 1965; Mossin, 1966) to a model including both liquidity level and liquidity risk. This model assumes a simple overlapping generation economy in which agents maximised their expected utility function in a one-period framework and introduce illiquidity costs by adding a per-share relative cost c_t^i. In this framework the standard CAPM translated into a CAPM in *net* (liquidity-cost adjusted: $r_j^{Net} = r_j - c_j$) returns for the imagined economy with illiquidity costs. By rewriting the standard one-beta CAPM in terms of gross returns Acharya and Pedersen (2005) obtain the L-CAPM:

$$\begin{aligned}
E_{t-1}(r_t^i - r_t^f) &= kE_{t-1}(c_t^i) + \lambda_{t-1}\frac{cov_{t-1}(r_t^i - c_t^i, r_t^M - c_t^M)}{var_{t-1}(r_t^M - c_t^M)} \\
&= kE_{t-1}(c_t^i) + \lambda_{t-1}\left[\frac{cov_{t-1}(r_t^i, r_t^M)}{var_{t-1}(r_t^M - c_t^M)} + \frac{cov_{t-1}(c_t^i, c_t^M)}{var_{t-1}(r_t^M - c_t^M)}\right. \\
&\left. - \frac{cov_{t-1}(r_t^i, c_t^M)}{var_{t-1}(r_t^M - c_t^M)} - \frac{cov_{t-1}(c_t^i, r_t^M)}{var_{t-1}(r_t^M - c_t^M)}\right] \\
&= kE_{t-1}(c_t^i) + \lambda_{t-1}\left[\beta_{i,t}^1 + \beta_{i,t}^2 - \beta_{i,t}^3 - \beta_{i,t}^4\right]
\end{aligned} \tag{1}$$

In addition to the traditional market beta, the L-CAPM introduces three new betas capturing different liquidity risks for an asset. These are: $\beta_{i,t}^2$ the commonality in illiquidity investors ask for a premium for holding a security that becomes illiquid when the market becomes illiquid, $\beta_{i,t}^3$ the security return sensitivity to market illiquidity, the risk to obtain low return when the market is illiquid and $\beta_{i,t}^4$ the sensitivity of the security's illiquidity to market returns, the risk to hold an illiquid asset in bad states of the market.

The risk price is given by $\lambda_{t-1} = E_{t-1}(r_t^M - c_t^M - r_t^f)$ and r_t^f is the risk free-rate. Superscripts i and M represent the security i and aggregate markets respectively. Thus c_t^i, c_t^M, r_t^i and r_t^M are, respectively, the portfolio illiquidity, market illiquidity, portfolio (gross) return and market (gross) returns. We depart from this model by restricting the risk premium (λ_t) to be constant (λ). This simplification is made to avoid identification problems and has also been adopted by Hagstromer et al., (2013). The scaling factor k refers to the holding period and is required to adjust the liquidity level premium to the number of times illiquidity costs are incurred.

3. Econometric specification of Conditional LCAPM

Our methodology can be summarised in five steps. Firstly, for each individual security *i* and at a *daily* frequency we estimate the bid-ask spread. Secondly, we form an equally-weighted market portfolio and sets of 25 test portfolios sorted in the basis of illiquidity. Thirdly, for each portfolio, we estimate the Local Linear Trend model with seasonal (LLT) and the Local Linear Trend model with common Stochastic Volatility and seasonal (LLTSV) model to extract, respectively, illiquidity innovations and stochastic volatility. Fourthly, using these innovations and stochastic volatility, we estimate at the daily frequency the time-varying betas through Dynamic Conditional Correlation (DCC) modelling. Fifthly, different DCC models specifications are considered based on the univariate conditional variance modelling framework. This results in several sets of time-varying betas that are ranked accordingly their power to explain cross-sections in portfolio returns. Finally we run cross-sectional regressions at the monthly frequency to test the model.

3.1. Modelling illiquidity via Unobserved Components Models

In this section we detail the two Unobserved Components models (also called Structural Models) employed to model the bid-ask spread. We choose to model bid-ask spread through four components: level, slope, seasonal and an autoregressive component. According to the efficient-market hypothesis, the liquidity of a stock should follow a martingale process because its level should incorporate all the available information. This level should therefore be unpredictable. To confirm with this view we incorporate a level component in the model, this component is modelled as a near unit-root process that represents the hidden level of the bid-ask spread. This component departs from a strict unit root process by the extra slope component we add it to it. This slope, or drift term, is the second component of our model and is itself specified as a random walk process. It aims to model the long-term trend component of the bid-ask spread. Due to competition among stock exchanges and to innovations in technology, we expect this term to be negative implying a long term increase in the market liquidity. Additionally we also incorporate a seasonal component intended to take into account the well-known day-of-the-week effects (Chordia at al. 2001) in liquidity series. Finally an AR(1) process is also considered as the fourth component, it aims to capture *temporary* bid-ask spread shocks caused by short term imbalance in liquidity demand and supply.

The four components are modelled as stochastic processes with their own error terms. Based on these four components we constructed two structural models: the Local Linear Trend model with seasonal (LLT) and the Local Linear Trend model with common Stochastic Volatility and seasonal (LLTSV)[3]. The LLTSV model has the additional feature of allowing for a time-varying conditional variance process, then volatility clustering is directly evaluated by the UC model (for the LLT model, it is evaluated indirectly by fitting a GARCH type of model to the residuals).

LLT Model

[3] We use the terminology "Local Linear Trend", but our model differs from the traditional LLT specification (see Harvey and Shephard 1993) since we add an autoregressive component and omit the irregular component.

The local linear trend model with seasonal and AR(1) component is formally given by:

$$y_t = \mu_t + \gamma_t + z_t \tag{2}$$

$$\mu_{t+1} = \mu_t + v_t + \xi_t, \qquad \xi_t : N(0,\sigma_\xi^2) \tag{3a}$$

$$v_{t+1} = v_t + \zeta_t, \qquad \zeta_t : N(0,\sigma_\zeta^2) \tag{3b}$$

$$\gamma_{t+1} = -\sum_{j=1}^{s-1} \gamma_{t+1-j} + \omega_t, \qquad \omega_t : N(0,\sigma_\omega^2) \tag{3c}$$

$$z_{t+1} = \phi \times z_t + \varepsilon_t, \qquad \varepsilon : N(0,\sigma_\varepsilon^2) \tag{3d}$$

The measurement equation (2) relates the observed variable y_t (the observed bid-ask spread) to the unobserved state variables where μ_t is the level, v_t the slope, z_t the autoregressive and γ_t the seasonal component, $t = 1,..., n$.
The transition equations (3a to 3d) model the dynamics of the unobserved components. Level, slope, autoregressive and seasonal component are treated as stochastic processes. The disturbance terms of the level, slope , seasonal and AR(1) components are respectively given by ξ_t, ζ_t, ω_t and ε_t. The condition $|\phi| < 1$ is imposed to ensure stationarity and identification of the short-term component

We use the Kalman filter[4] to obtain an estimate of the conditional distribution of the state variables at $t+1$ given the past data *only*, this removes the aforementioned look-ahead bias whose AR models suffer. The error in the prediction $e_t = y_t - E(y_t / Y_{t-1})$ and its variance $F_t = Var(e_t | Y_{t-1})$, Y_{t-1} information available at time t-1, are easily computed as a by-product of the Kalman Filter. These two parameters are employed to compute, for a given set of unknown parameters Ψ (disturbances variances and state parameters), the value of the log-likelihood function:

$$logL(\Psi) = -\frac{n}{2} log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left(logF_t + \frac{e_t^2}{F_t}\right)$$

Finally, we estimate the unknown parameters by maximising the log-likelihood function using the Broyden-Fletcher-Goldfarb-Shannon (BFGS) algorithm. Estimation is performed with the Ssfpack package (Koopman et al., 1999) [5][6].

LLT with Stochastic Volatility (LLTSV)

We also consider an extension of the previous model by adding a stochastic *common*

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[4] Derivation of the Kalman Filter is given in the appendix.
[5] See Commandeur and Koopman (2007) and Pelagatti (2011) for a gentle introduction to this package.
[6] Comparison of forecast accuracy between the LLT model and standard autoregressive models (i.e. the framework commonly used in the literature) is available from the authors on request.

variance component to the model. This is motivated by the fact that the conditional variance of liquidity times series is known to be time-varying and that volatility clustering exists. We employ the model proposed by Koopman and Bos (2004), they present an innovative way to combine a linear state space model (as the LLT model) with the stochastic volatility model.

The combined LLT trend model with stochastic volatility (LLTSV model) is formally given by:

$$y_t = \mu_t + \gamma_t + z_t \tag{4}$$

$$\mu_{t+1} = \mu_t + v_t + q_1\varepsilon_{1t} \tag{5a}$$

$$v_{t+1} = v_t + q_2\varepsilon_{2t} \tag{5b}$$

$$\gamma_{t+1} = -\sum_{j=1}^{s-1}\gamma_{t+1-j} + q_3\varepsilon_{3,t} \tag{5c}$$

$$z_{t+1} = \phi z_t + q_4\varepsilon_{4t} \tag{5d}$$

where the disturbance vector ε_t is: $\varepsilon_t : N(0, \sigma_t^2 I_4)$

We consider the simplest stochastic volatility model given by

$$\sigma_t^2 = exp(h_t)$$

$$h_{t+1} = (1-\phi)d + \phi h_t + \sigma_\eta \eta_t, \qquad \eta_t : N(0,1) t = 1, \ldots, n, \; 0 \le \phi < 1 \text{ and } \sigma_t > 0.$$

The disturbances ε_t and the disturbance η_t are assumed to be uncorrelated.

The process of the stochastic variable h_t is itself treated in state space form (see Koopman and Bos, (2004) for details about the system of matrices). Thus the LLTSV is a combination of two linear state space models. In terms of forecasting, Koopman and Bos (2004) show that the LLTSV model outperforms the LLT model because it increases the variability of the exponential weights that depend on the overall value of the conditional variance σ_t.

Estimation of the LLTSV model is far more complex than for the LLT model. The prediction error (v_t) cannot be extracted directly since σ_t is used in the Kalman Filter equations but is not known. The estimation procedure proposed by Koopman and Bos (2004) consists in approximating the true likelihood via averages of simulations from an approximating model. This procedure, implemented in the SSFSV package[7], is considerably CPU and time consuming, particularly for one-step-ahead forecasts. Consequently, we did not employ a one-step-ahead forecasting framework but a simplified approach by estimating the model in a single shot[8]. It worth noting that by doing so we re-introduce the look-ahead bias previously mentioned (only for the LLTSV model).

3.2. Estimating the Time Varying Beta

In this part we present the methodology employed to obtain the time-varying betas. The time-varying correlations are obtained via the Dynamic Conditional Correlation (DCC) model of

[7] Available at http://personal.vu.nl/c.s.bos/publications/ssfsv.htm.

[8] The stochastic volatility part is estimated using the whole sample instead of being based on a moving-block as it should be.

Tse and Tsui, (2002). More precisely, we consider a four-dimensional vector:

$$Y_t = (c_t^i, c_t^M, r_t^i, r_t^M), \qquad t = 1:T$$

$$Y_t : N(0, H_t)$$

where c_t^i (c_t^M) and r_t^i (r_t^M) denote respectively the portfolio (market) liquidity innovations (obtained from unobserved models) and returns. They are zero mean processes with $var(Y_t \mid I_{t-1}) = H_t$, where the conditional variance matrix H_t follows the DCC specification of:

$$H_t = D_t R_t D_t = \rho_{ijt}\sqrt{h_{iit} h_{jjt}}$$

$$D_t = diag(H_{11t}^{1/2}, \ldots, H_{NNt}^{1/2})$$

$$R_t = (1 - a - b)R + bR_{t-1} + a\Phi_{t-1}$$

where a and b are non-negative parameters satisfying $a + b < 1$. R is a symmetric NXN (4×4) positive definite parameter matrix with $\rho_{ii} = 1$ and Φ_{t-1} is the NXN sample correlation matrix derived from the standardised residuals. This model imposes GARCH type dynamics on the conditional correlation matrix R_t.

D_t is obtained in a pre-estimation via two univariate conditional variance models: the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model (Bollerslev, 1986) with 1 lag, and the Beta-T-Garch (Harvey and Chakravarty, 2008) a special case of the Generalised Autoregressive Score (GAS) model (Creal et al, 2013). Daily liquidity series exhibit numerous jumps probably due to noise, these jumps affect future volatility less than what standard volatility models would predict. To deal with this issue, GAS models are ideal candidates because they can lower the effect of jumps on the conditional variance process. In this model, the conditional variance is derived from the conditional score of the innovation distribution with respect to the second moment. The novelty is to link the shape of the conditional variance density directly to the dynamics of conditional variance. The specification of the GAS(1,1) equation is given by:

$$\sigma_t^2 = w + \alpha_1 u_{t-1}\sigma_{t-1}^2 + \phi_1 \sigma_{t-1}^2$$

$$u_t = \frac{(v+1)z_t^2}{v - 2 + z_t^2} \qquad z_t : t(0,1,v)$$

where u_t is a a rescaled conditional score and z_t are *standardised* errors. In this model, the variance is driven by the conditional score of the last observation, and implies a fast adjustment of parameters following new observations.

Once the correlation matrix R_t and the conditional standard deviations D_t are estimated the time-varying betas are easily estimated as a by-product using the following formula (applied in this case to the second beta):

$$\beta_t^2 = \frac{cov_t(c_t, c_t^M)}{var_t(r_t^M - c_t^M)} = \frac{\rho_t \sqrt{h_t^{c_t}} \sqrt{h_t^{c_t^M}}}{h_t^{r_t^M - c_t^M}}$$

where $h_t^{c_t^M}, h_t^{c_t^M}$ and $h_t^{r_t^M - c_t^M}$ are, respectively, the conditional variance of the portfolio illiquidity innovations[9], market illiquidity innovations and the series defined as market returns minus market illiquidity innovations. The three other betas are obtained similarly. The proposed methodology is similar to that proposed by Bollerslev et al. (1988) except that we have multiple betas.

The DCC model is estimated via the MLE method by a two step approach, parameters of the univariate conditional variance processes are estimated in the first step, then conditional correlation parameters are estimated in a second step based on standardised series. The G@rch package (Laurent and Peters, 2002) is employed to produce estimates.

As the conditional variances of the returns ($h_t^{r_t}, h_t^{r_t^M}$) are obtained via two GARCH type of models (GARCH or GAS) and the conditional variance of illiquidity ($h_t^{c_t}, h_t^{c_t^M}$) is also obtained in two ways (LLT or LLTSV), we construct four sets of time-varying betas leading us to test four different L-CAPM models, labelled models I to IV and summarized in Table 1.

Table 1: Four estimation models of time-varying betas. Models differ in the way the conditional variance of illiquidity or returns are obtained.

Model	Name	UC	h_{ijt} Illiquidity	h_{ijt} returns
I	DCC-GARCH	LLT	GARCH	GARCH
II	DCC-GAS	LLT	GAS	GAS
III	DCC-STOCH GARCH	LLTSV	Stochastic Vol.	GARC
IV	DCC-STOCH GAS	LLTSV	Stochastic Vol.	GAS

4. Empirical study

In this section, parameters of the unobserved components models and the time-varying betas are presented. Next, we estimate and test the L-CAPM. Finally we investigate the composition of the illiquidity premium.

4.1. Data

Daily data is extracted from the Bloomberg database and contains all common firms listed on the NASDAQ from 1 January 2006 to 31 December 2014 and available via the database[10]. To build a reliable sample, we apply the following screening procedure. For a stock to be included in the sample, it should have at least 250 days with data over the entire sample period. To prevent the influence of extreme stocks, we drop, for each year, all stock whose the

[9] the subscript i has been dropped for the ease of exposure.

[10] We select the NASDAQ because it allows comparison with previous studies based on the same index.

current year stock price is less than or equal to 1 dollar or greater than or equal to 1000 dollars. Stocks are also required to have at least 15 valid daily observation in a given month and the bid-ask spread is capped at 40 %. Finally, to avoid the survivor-ship bias, we retain all data for dead stocks in the sample and we consider a -30 % return when delisting occurs[11]. Table 2 shows the number of stocks included in the analysis on a yearly basis. For each stock, the daily data set contains last price, closing bid, closing ask, capitalisation, price to book ratio and turnover ratio. We employ the (daily) relative *realised* bid-ask spread proxy (Goyenko et al., 2009) because it is a per-share measure of transaction cost and it fits directly to the theoretical model, it is defined as:

$$\text{Realised\_Spread}_{i,t} = \frac{|P_{i,t} - MQ_{i,t+}|}{MQ_{i,t+}} \qquad (6)$$

where $P_{i,t}$ refers to the price of the last trade and $MQ_{i,t+}$ is the mid-quote just after this trade for stock i and day t. The closing bid and ask price used to compute the mid-quote are necessarily quoted after the timing of the last trade but our dataset does not provide information about this delay. However, we can reasonably assume that it is homogeneous across stocks because bid and ask quotes are continuously updated after a trade is triggered. As far as we read, it is the first study that directly uses a per-share measure of transaction cost [12] to test the L-CAPM.

In line with common practice in the literature (Acharya and Pedersen, 2005; Hagstromer et al., 2013), we perform our analysis on a portfolio basis. Thus, we split the whole sample into 25 annual (equally weighted) rebalanced portfolio, sorted on the basis of liquidity: portfolio 1 being composed of the highest liquid stocks (lowest spread) and portfolio 25 the lowest liquid stocks (highest spread). We use equally weighted portfolio instead of value-weighted portfolios because value-weighted portfolio tend to overestimate the importance of large liquid stocks.

Table 2: Number of stocks per year.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Nb. Of Stocks	2534	2760	3004	2258	2095	2510	2502	2386	2419	2422

4.1. Unobserved Components Analysis

The parameter estimates of the LLT model and LLTSV model are presented in Table 3 and Table 4. Regarding the LLT model, the estimates of the standard deviation of the slope disturbances have nearly zero values. This indicates that the slope component treated as stochastic process in the current implementation could be alternatively considered as a deterministic processes (i.e. with a constant drift term) without problematic loss of information. State estimates of the slope components (not reported) are found to be negative and significant for some periods for all series, which shows that there is an increasing trend of liquidity during the whole survey period.

[11] This procedure has been adopted by Hagstromer et al. (2013)
[12] Not a *proxy* such as the effective tick proxy of Holden (2009) or the Amihud proxy.

The estimates of the variance of the seasonal disturbances take very low values, indicating that seasonality in liquidity is highly predictable. Finale state estimates of the seasonality are presented in Table 5. We observe a clear positive seasonal effect on Friday for most of the portfolios (as in Chordia et al. 2001). This is probably due to the fact that agents anticipate the weekend effect, a period with higher risk and uncertainty. In contrast, Wednesday and Thursday have a negative effect on the bid-ask spread. This initial finding suggests the integration of a slope and seasonal component in liquidity forecasting models since these components are highly predictable. As far as we read, they have never been considered in testing the L-CAPM model.

In order to analyse the relative importance of each variance we compute the ratio of each variance on the total variance of a given portfolio (given in parenthesis in tables 3 and 4). We observe that the the relative importance of slope (AR) disturbances increases (decreases) with the level of illiquidity. This important finding indicates that the bid-ask spread process of a liquid portfolio is more impacted by short term shocks than that of illiquid portfolios. In other words, liquid stocks have more risk of suffering a temporary shock than low liquid stocks. In contrast, the variability of the bid-ask spread of low liquid stocks is more impacted by structural (i.e. permanent) shocks. This relationship is also confirmed for the LLTSV model (Table 4).

Regarding the parameter estimates of the LLTSV model, the autoregressive parameter ϕ of the common stochastic variance is estimated in the range [0.929:0.989] and is not correlated with illiquidity. These high values demonstrate that the variance process of bid-ask spread series has a strong persistence, irrespective of the liquidity level of the series studied. Periods of high liquidity volatility have a long-lasting effect. We also found that for both models (LLT and LLTSV), the AR component is the component which has the most important disturbance term ($\hat{\sigma}_{\varepsilon}$ in table 3 and $\sigma^2_{q_4}$ in table 4). This is strong evidence that bid-ask spread series are heavily impacted by temporary shocks. Undoubtedly, this finding must be considered in empirical researches, specifically when it is necessary to differentiate between liquidity variation that is due to a change in the intrinsic feature of a stock (level component) or that is due to temporary imbalance in liquidity supply and demand (AR component). Values of the AR(1) parameter are found to be in the range [0.14:0.28] for the LLT model and [0.161:0.229] for the LLTSV model. This implies that temporary shocks have a non-negligible impact on the mean process of the bid-ask spread during a period of around 4 to 6 days. Interestingly we found, for both models, that the market portfolio has the highest estimated AR(1) parameter, systematic (market-wide) shocks have a longer-lasting effect on the liquidity of the market than idiosyncratic shock.

To make the model more meaningful, Figures 1 and 2 display the estimated unobserved components of the bid-ask spread of the market for, respectively, the LLT and the LLTSV model. The change in bid-ask spread over time is reflected by the estimated level component. We observe that the level component is smoother than the raw bid-ask spread series, due to the fact that the variance of the error term of this component is low and to extraction of the AR(1) component. The daily seasonal components exhibit a strong similar periodicity pattern, confirming our previous results. The common variance component[13] (lower left panel of Figure 1) exhibits one period of high volatility: from September to December 2008 corresponding to

[13] scaled by the variance of the disturbance term of the AR component.

the Global financial crisis with the Lehman Brothers bankruptcy.

To summarise this paragraph, we found that bid-ask spread series exhibit a strong daily periodicity, we observe an overall increase in the market liquidity during the period studied, we find that the bid-ask spread process of low liquid stocks is relatively more impacted by permanent shocks than the bid-ask spread process of liquids stocks, and we found a strong persistence in the volatility of liquidity. The main source of variation is due to temporary liquidity imbalances (AR(1) component). Finally we also found that (temporary) systematic shocks on bid-ask spreads have a longer lasting effect than specific (to portfolios) temporary shocks.

Figure 1: LLT Unobserved Components. This figures reports the evolution of the market's bid ask spread (upper right panel), level component (upper left), seasonal component (lower left) and AR component (lower right) for the LLT model. The seasonal component is displayed for the last 5 weeks of the sample.

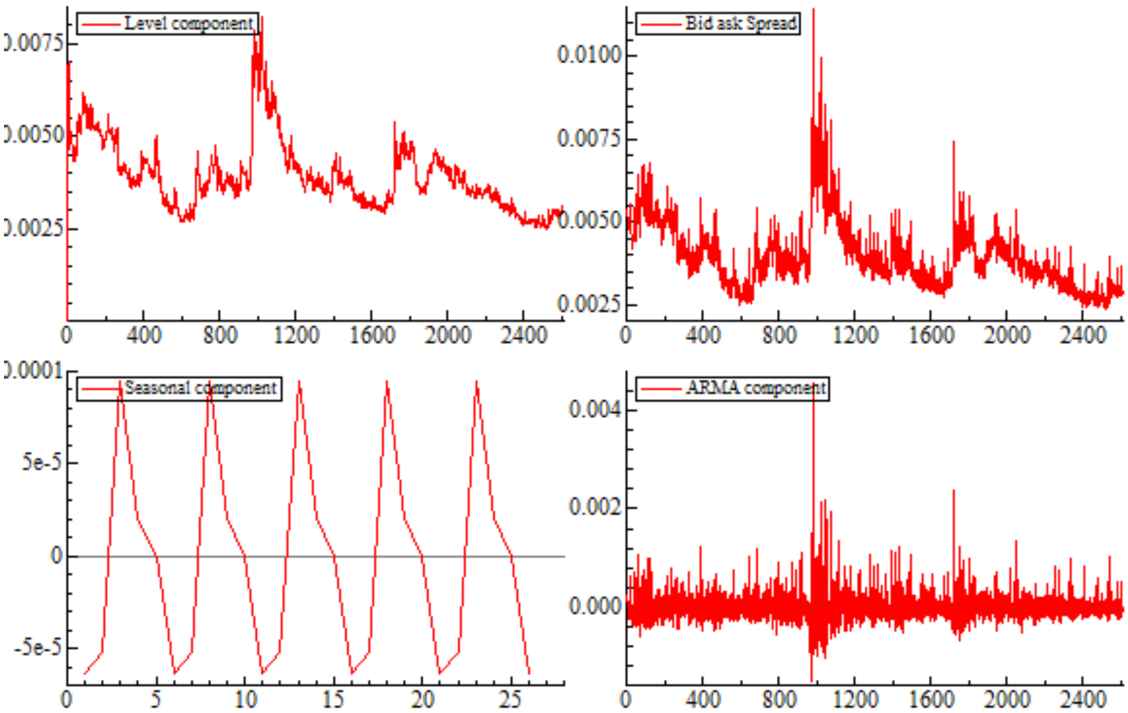

Figure 2: LLTSV Unobserved Components. This figures reports the evolution of level component (upper left pannel), Seasonal component (upper right), AR component (lower right) and common variance ($q_4\sigma_t^2$) (lower right) for the LLTSV model. The seasonal component is displayed for the last 4 months of the sample.

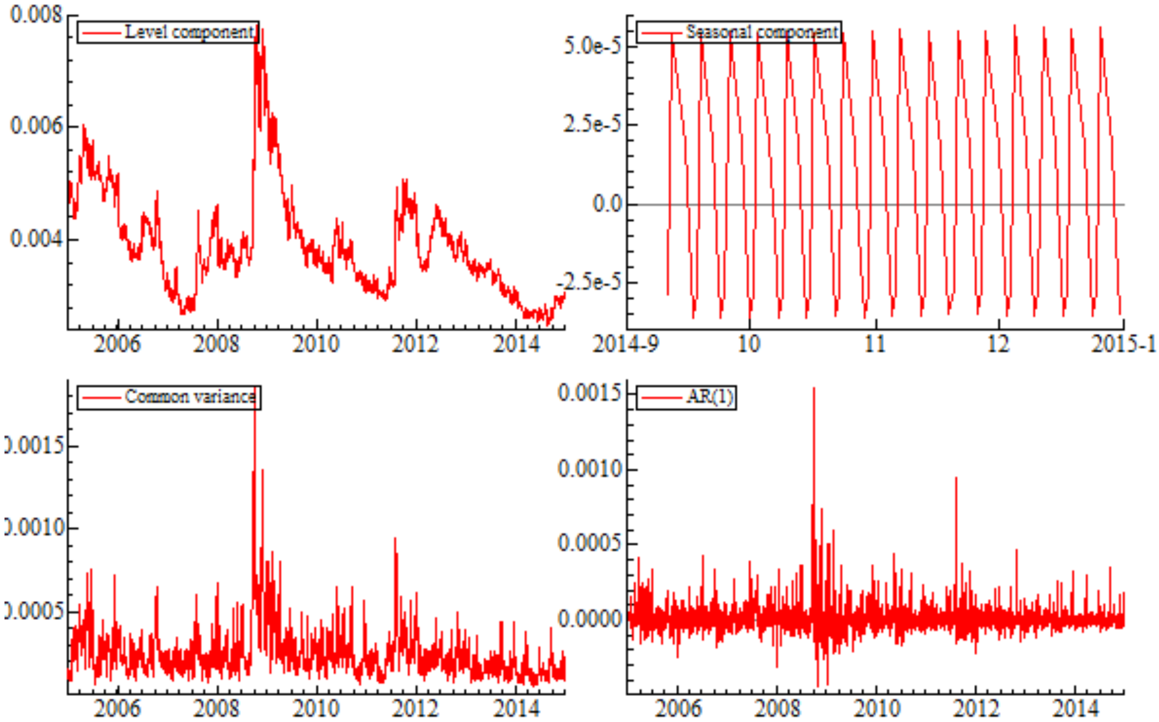

Table 3: LLT parameters estimates. This table reports estimates of the standard deviations of the level, slope, seasonal and autoregressive state disturbances. The respective v-ratio are given in parenthesis.

Portfolios	$\hat{\sigma}_{\xi}$ (Level)	$\hat{\sigma}_{\zeta}$ (Slope)	$\hat{\sigma}_{w}$ (Seasonal)	$\hat{\sigma}_{\varepsilon}$ (AR)	AR(1) ϕ
Port 1	8.10e-06(0.10)	1.10e-11(0.00)	1.94e-07(0.00)	6.91e-05(0.89)	0.23
Port 2	1.04e-05(0.12)	1.89e-11(0.00)	3.41e-07(0.00)	7.37e-05(0.87)	0.24
Port 3	1.34e-05(0.13)	3.55e-09(0.00)	2.84e-07(0.00)	8.99e-05(0.87)	0.23
Port 4	1.54e-05(0.13)	3.56e-09(0.00)	3.17e-07(0.00)	9.97e-05(0.86)	0.19
Port 5	1.78e-05(0.11)	4.07e-11(0.00)	5.09e-13(0.00)	1.42e-04(0.89)	0.16
Port 6	2.02e-05(0.15)	1.05e-11(0.00)	2.29e-07(0.00)	1.18e-04(0.85)	0.24
Port 7	2.00e-05(0.13)	5.50e-12(0.00)	4.64e-07(0.00)	1.36e-04(0.87)	0.23
Port 8	2.66e-05(0.15)	2.46e-11(0.00)	2.74e-07(0.00)	1.48e-04(0.85)	0.21
Port 9	3.15e-05(0.16)	9.52e-12(0.00)	4.05e-07(0.00)	1.66e-04(0.84)	0.20
Port 10	3.79e-05(0.15)	1.87e-11(0.00)	3.78e-07(0.00)	2.18e-04(0.85)	0.14
Port 11	3.69e-05(0.16)	3.30e-11(0.00)	9.33e-07(0.00)	1.97e-04(0.84)	0.21
Port 12	4.72e-05(0.17)	2.11e-11(0.00)	6.39e-07(0.00)	2.24e-04(0.82)	0.21
Port 13	4.98e-05(0.16)	4.35e-10(0.00)	8.08e-10(0.00)	2.52e-04(0.84)	0.25
Port 14	6.23e-05(0.18)	5.83e-15(0.00)	1.42e-09(0.00)	2.89e-04(0.82)	0.16
Port 15	7.76e-05(0.18)	3.60e-13(0.00)	2.49e-10(0.00)	3.48e-04(0.82)	0.23
Port 16	1.06e-04(0.23)	3.16e-14(0.00)	8.90e-10(0.00)	3.65e-04(0.77)	0.23
Port 17	1.07e-04(0.20)	4.07e-10(0.00)	3.01e-06(0.01)	4.25e-04(0.79)	0.25
Port 18	1.33e-04(0.20)	1.49e-09(0.00)	1.84e-06(0.00)	5.27e-04(0.80)	0.24
Port 19	1.66e-04(0.20)	9.31e-11(0.00)	2.04e-06(0.00)	6.58e-04(0.80)	0.18
Port 20	2.09e-04(0.22)	8.35e-11(0.00)	2.57e-06(0.00)	7.20e-04(0.77)	0.19
Port 21	2.66e-04(0.23)	2.37e-11(0.00)	7.98e-06(0.01)	8.75e-04(0.76)	0.27
Port 22	2.78e-04(0.21)	2.16e-21(0.00)	1.16e-09(0.00)	1.03e-03(0.79)	0.21
Port 23	3.57e-04(0.22)	0.00e+00(0.00)	1.21e-09(0.00)	1.24e-03(0.78)	0.20
Port 24	4.37e-04(0.21)	3.35e-10(0.00)	2.27e-10(0.00)	1.61e-03(0.79)	0.28
Port 25	1.00e-03(0.30)	5.66e-11(0.00)	7.14e-06(0.00)	2.37e-03(0.70)	0.25
Market	1.08e-04(0.25)	1.64e-09(0.00)	1.07e-08(0.00)	3.16e-04(0.75)	0.29

Table 4: LLTSV parameters estimates. This table reports estimates of the standard deviation of the level, slope, seasonal and irregular state disturbances. The respective v-ratio are given in parenthesis.

Portfolios	$\sigma^2_{q_1}$ (Level)	$\sigma^2_{q_2}$ (Slope)	$\sigma^2_{q_3}$ (Seasonal)	$\sigma^2_{q_4}$ (AR(1))	σ^2_{η}(CSV)	ϕSV	AR(1)
Port 1	5.01e-04(.10)	2.18e-10(.00)	3.38e-05(.01)	4.34e-03(.89)	2.73e-01	0.975	0.191
Port 2	6.87e-04(.12)	6.71e-09(.00)	2.92e-05(.01)	4.88e-03(.87)	2.15e-01	0.984	0.221
Port 3	8.90e-04(.12)	6.55e-07(.00)	4.08e-05(.01)	6.20e-03(.87)	1.96e-01	0.985	0.189
Port 4	1.06e-03(.12)	8.30e-07(.00)	4.86e-05(.01)	7.44e-03(.87)	1.55e-01	0.989	0.166
Port 5	1.28e-03(.13)	2.05e-10(.00)	6.56e-05(.01)	8.75e-03(.87)	2.48e-01	0.977	0.201
Port 6	1.32e-03(.13)	1.51e-09(.00)	4.31e-05(.00)	9.04e-03(.87)	1.46e-01	0.989	0.195
Port 7	1.33e-03(.12)	5.36e-10(.00)	7.36e-05(.01)	9.63e-03(.87)	2.19e-01	0.980	0.214
Port 8	1.67e-03(.14)	1.05e-10(.00)	5.47e-05(.00)	1.06e-02(.86)	2.33e-01	0.975	0.161
Port 9	1.95e-03(.14)	5.31e-11(.00)	5.83e-05(.00)	1.22e-02(.86)	2.38e-01	0.975	0.185
Port 10	2.38e-03(.15)	3.08e-12(.00)	1.41e-04(.01)	1.32e-020.84)	3.37e-01	0.953	0.179
Port 11	2.34e-03(.13)	5.07e-11(.00)	1.67e-04(.01)	1.52e-02(.86)	2.29e-01	0.973	0.205
Port 12	2.99e-03(.14)	1.54e-10(.00)	2.29e-05(.00)	1.76e-02(.85)	2.66e-01	0.960	0.186
Port 13	3.68e-03(.15)	4.90e-11(.00)	9.05e-05(.00)	2.00e-02(.84)	2.49e-01	0.964	0.221
Port 14	3.88e-03(.15)	2.75e-11(.00)	1.24e-04(.00)	2.21e-02(.85)	3.04e-01	0.950	0.194
Port 15	5.61e-03(.18)	5.40e-10(.00)	8.82e-05(.00)	2.58e-02(.82)	3.50e-01	0.929	0.194
Port 16	6.21e-03(.17)	2.93e-11(.00)	2.23e-04(.01)	3.02e-02(.82)	2.64e-01	0.952	0.190
Port 17	6.49e-03(.16)	8.70e-13(.00)	3.40e-04(.01)	3.49e-02(.84)	2.89e-01	0.940	0.211
Port 18	7.66e-03(.15)	2.36e-10(.00)	2.19e-04(.00)	4.27e-02(.84)	3.27e-01	0.926	0.180
Port 19	1.07e-02(.18)	4.04e-14(.00)	6.53e-05(.00)	5.03e-02(.82)	4.15e-01	0.883	0.170
Port 20	1.25e-02(.17)	9.30e-10(.00)	1.10e-04(.00)	6.07e-02(.83)	3.13e-01	0.926	0.225
Port 21	1.58e-02(.18)	2.65e-10(.00)	3.63e-04(.00)	7.36e-02(.82)	2.79e-01	0.942	0.229
Port 22	1.93e-02(.18)	1.39e-10(.00)	4.97e-11(.00)	8.56e-02(.82)	2.73e-01	0.943	0.192
Port 23	2.74e-02(.21)	2.73e-15(.00)	3.27e-15(.00)	1.05e-01(.79)	1.57e-01	0.980	0.175
Port 24	3.39e-02(.20)	2.45e-11(.00)	2.58e-09(.00)	1.37e-01(.80)	2.12e-01	0.960	0.219
Port 25	5.88e-02(.22)	4.18e-10(.00)	5.05e-04(.00)	2.08e-01(.78)	1.56e-01	0.977	0.226
Market	6.61e-03(.23)	3.03e-09(.00)	1.28e-04(.00)	2.21e-02(.77)	4.99e-01	0.885	0.314

Table 5: Final state estimates of the seasonal component. This table presents final state estimate of the seasonal component. Stars indicate rejection of the null hypothesis at *** 0.01; **0.05 and *0.1 confidence level.

Portfolios	Monday	Tuesday	Wednesday	Thursday	Friday
Port 1	-4.78e-06	5.54e-07	-3.04e-06	-2.47e-06	9.74e-06**
Port 2	-3.03e-06	-2.85e-06	-3.63e-07	-2.38e-06	8.62e-06
Port 3	-3.92e-06	-1.90e-06	-4.36e-06	-4.06e-07	1.06e-05*
Port 4	-3.50e-06	-3.05e-06	-4.26e-06	-3.55e-06	1.44e-05**
Port 5	-6.42e-06	5.63e-06	-1.03e-05*	-6.92e-06	1.80e-05***
Port 6	-4.05e-06	-4.41e-06	-7.74e-06	-3.50e-06	1.97e-05***
Port 7	-5.04e-06	-6.82e-06	-6.31e-06	-5.14e-06	2.33e-05***
Port 8	-3.87e-06	-5.26e-06	-8.55e-06	-7.69e-06	2.54e-05***
Port 9	-5.94e-06	-1.09e-05	-8.19e-06	-4.20e-06	2.92e-05***
Port 10	-8.33e-06	-5.82e-06	-1.62e-05	-8.00e-06	3.84e-05***
Port 11	-4.77e-06	-1.31e-05	-6.69e-06	-7.93e-06	3.25e-05**
Port 12	-1.13e-05	-1.17e-05	-1.63e-05	-6.66e-06	4.60e-05***
Port 13	-1.11e-05	-3.12e-06	-2.42e-05**	-1.50e-05	5.35e-05***
Port 14	-1.94e-05*	-8.66e-06	-2.55e-05**	-2.32e-05**	7.67e-05***
Port 15	-2.53e-05*	-6.05e-06	-3.30e-05**	-1.63e-05	8.08e-05***
Port 16	8.38e-06	-1.75e-05	-5.15e-05***	-3.28e-05**	9.35e-05***
Port 17	-5.22e-05	-3.01e-05	-1.04e-05	-3.10e-05	1.24e-04***
Port 18	-4.52e-06	-2.89e-05	-6.37e-05*	-7.90e-05**	1.76e-04***
Port 19	2.66e-05	-2.32e-05	-8.49e-05**	-1.10e-04***	1.91e-04***
Port 20	1.52e-05	-5.66e-05	-1.16e-04**	-9.46e-05**	2.52e-04***
Port 21	1.16e-05	-4.95e-05	-1.33e-04	-1.08e-04	2.79e-04***
Port 22	3.69e-05	-6.93e-05*	-1.04e-04***	-5.32e-05	1.89e-04***
Port 23	8.35e-05*	-4.62e-05	-1.46e-04***	-1.26e-04***	2.35e-04***
Port 24	1.02e-04*	-3.92e-05	-1.32e-04**	-2.35e-04***	3.03e-04***
Port 25	1.26e-04	2.79e-05	-3.02e-04**	-2.67e-04*	4.15e-04***
Market	2.05e-05*	-1.93e-07	-6.29e-05***	-5.20e-05***	9.47e-05***

4.2 Conditional betas

Following Acharya and Pedersen (2005) and Hagstromer et al. (2013), the pricing analysis takes place at a monthly frequency. In consequence, for each month and portfolio we average the daily liquidity betas to obtain monthly liquidity beta estimates. We also compute the unconditional betas to allow comparison (table 10), they are simply computed as the coefficient of unconditional covariance over unconditional variance terms. The four time-varying betas are computed as explained in subsection 3.2. Table 6 displays descriptive statistics obtained using model I.

Contrary to previous studies, we do not observe (table 6) a clear positive relationship between returns (r) and illiquidity level (bid-ask), at first sight it seems to indicate that investors do not require compensation for illiquidity. However this fact can be caused by a higher sensitivity of liquid portfolio to the market risk. The value of the standard market beta (β_1) is clearly negatively correlated with illiquidity. Illiquidity portfolios are more exposed to the idiosyncratic risk: portfolio 1 has an average β_1 value of 1.018 and portfolio 25 a value of 0.4341. We also found that the portfolio return sensitivity to market illiquidity (β^3), and the portfolio illiquidity sensitivity to market returns (β^4) have negative average coefficients. Given that they enter the pricing equation with a negative sign (see equation 1), this shows that they effectively correspond to a discount to hold a portfolio with high return when the liquidity of the market is low and to get a liquid portfolio when market returns are low. As expected, illiquid portfolios tends to have a smaller capitalisation (SIZE) and Price to Book ratio (PBR).

Regarding liquidity risks, we find that illiquid stocks have a high liquidity risk related to β^2 and β^4 - they have large value of β^2 and large negative values of β^4. The positive correlation between the coefficient of β^2 and illiquidity demonstrates that the liquidity of illiquid portfolios is more impacted by changes in the market liquidity than top liquid portfolios. In others word, the commonality in illiquidity (β^2) is higher for low liquid portfolios than for top liquid portfolios. The risk of having a decrease in liquidity during a liquidity crisis period is higher for illiquid portfolios than for liquid portfolios. The observed positive relation between coefficient β^4 and illiquidity indicates also that the risk of experiencing a decrease in the liquidity level of a portfolio when the market return is low is more important for low liquidity portfolios.

The previous observations are similar to that made by Acharya and Pedersen (2005) and Hagstromer et al. (2013) by associating a higher liquidity risk to illiquid portfolio. However the behaviour of β^3,the portfolio return sensitivity to market illiquidity, is different from that observed by the previously mentioned studies. Interestingly, we found that the associated liquidity risk is lower for illiquid stocks - they have smaller negative values of β^3. It indicates that the returns of illiquid portfolios are less sensitive to market illiquidity. This new finding will be investigated later on in subsection 4.3.

Table 7 reports the time-series average correlations of betas, we observe that betas are correlated and this is taken into account in the cross-section analysis part. Then the different liquidity risks are correlated. This multicollinearity problem has also been documented for the Unconditional L-CAPM in Acharya and Pedersen (2005).

The main results of this paragraph are that liquidity risks related to β_2 and β_4 are positively correlated to illiquidity and that the liquidity risk attached to β_3 is *negatively* correlated to illiquidity. Thus, the liquidity risk is not always associated with a higher illiquidity level.

Table 6: Properties of illiquidity Portfolio. This table reports descriptive statistics of portfolios. The *conditional* beta are represented as time-series average for each portfolio and indicating by $E(\beta_t^n)$ where n refers to the corresponding beta. Bid-ask and r are the monthly average of illiquidity (computed via Eq. 6) and gross returns. Portfolio 1 is the highest liquid portfolio (with smallest spread) and portfolio 25 is the lowest liquid portfolio. Values of the four beta and the bid ask spread have been pre-mutliplied by 100 to ease reading. Betas correspond to those obtained from model I.

Port.	Bid-ask	r	$\sigma(r)$	$E(\beta_t^1)$	$E(\beta_t^2)$	$E(\beta_t^3)$	$E(\beta_t^4)$	SIZE	PBR
1	.037	0.016	0.011	101.804	0.232	-0.965	-0.264	9.581	3.694
2	0.047	0.017	0.012	116.309	0.283	-1.082	-0.111	8.584	3.062
3	0.055	0.016	0.013	119.385	0.282	-1.097	-0.350	8.205	2.837
4	0.063	0.016	0.013	122.432	0.286	-1.076	-0.287	7.749	2.639
5	0.071	0.016	0.014	128.083	0.327	-1.101	-0.261	7.345	2.371
6	0.079	0.015	0.014	127.714	0.315	-1.142	-0.468	7.269	2.211
7	0.088	0.015	0.014	127.576	0.331	-1.115	-0.608	7.125	2.078
8	0.098	0.013	0.014	127.450	0.340	-1.124	-0.651	6.791	1.937
9	0.109	0.015	0.015	135.197	0.345	-1.083	-0.555	6.658	1.895
10	0.124	0.012	0.014	132.213	0.349	-1.016	-0.411	6.617	1.757
11	0.140	0.015	0.014	128.257	0.371	-1.007	-0.825	6.315	1.746
12	0.161	0.012	0.014	128.037	0.363	-1.019	-0.622	6.478	1.719
13	0.184	0.011	0.014	121.896	0.381	-0.913	-0.796	6.231	1.728
14	0.214	0.013	0.013	115.728	0.328	-0.890	-0.849	6.448	1.693
15	0.252	0.012	0.012	109.423	0.349	-0.973	-0.994	6.048	1.609
16	0.299	0.009	0.011	99.725	0.325	-0.959	-0.824	6.183	1.581
17	0.354	0.005	0.011	90.929	0.282	-0.969	-0.795	6.509	1.612
18	0.422	0.011	0.010	84.275	0.256	-0.832	-0.636	5.631	1.592
19	0.505	0.011	0.009	75.346	0.517	-0.727	-1.475	5.319	1.475
20	0.605	0.012	0.008	61.854	0.734	-0.668	-1.707	5.015	1.422
21	0.733	0.010	0.008	56.451	0.605	-0.675	-1.844	4.602	1.347
22	0.895	0.013	0.007	45.674	0.679	-0.663	-2.069	4.367	1.251
23	1.121	0.012	0.007	40.497	0.520	-0.599	-1.332	4.018	1.267
24	1.458	0.015	0.007	43.138	0.720	-0.523	-2.709	3.987	1.228
25	2.368	0.026	0.008	43.412	1.155	-0.614	-2.182	3.569	1.369

Table 7: Average beta correlations for illiquidity portfolios. This table reports the time-series average beta correlation for the 25 portfolios. Beta are obtained via Model I.

	β_1	β_2	β_3	β_4
β_1	1	0,206	-0,332	-0,002
β_2		1	-0,567	-0,482
β_3			1	0,471
β_4				1

4.3 Price of Liquidity Risk

In this paragraph, we estimate and test the L-CAPM. Seven cross-sectional regressions are estimated on the 25 portfolios. The seven cross-sectional regressions are designed in such a way that we can differentiate the various liquidity risks. We include size ($S_{i,t}$) and price to book ratio ($P_{i,t}$) as control variables because they are known to be determinant of returns. The regressions are formally presented bellow:

[Eq1]: $(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\alpha_s S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

[Eq2]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_{Net}\beta^{Net}_{i,t}+\alpha_s S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

[Eq3]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\lambda_2\beta^2_{i,t}+\lambda_3\beta^3_{i,t}+\lambda_4\beta^4_{i,t}+\alpha_1 S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

[Eq4]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\lambda_2\beta^2_{i,t}+\alpha_s S_{i,t}+\alpha_2 P_{i,t}+\varepsilon_{i,t}$

[Eq5]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\lambda_3\beta^3_{i,t}+\alpha_s S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

[Eq6]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\lambda_4\beta^4_{i,t}+\alpha_s S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

[Eq7]: $E(r_{i,t}-r_{t,f})-k_{i,t}E(c_{i,t})=\alpha+\lambda_1\beta^1_{i,t}+\lambda_{Liq}\beta^{Liq}_{i,t}+\alpha_s S_{i,t}+\alpha_p P_{i,t}+\varepsilon_{i,t}$

Equation 1 refers to a modified version of the *conditional* CAPM model[14]. Equation 2 corresponds to the theoretical pricing model, in which both liquidity level and liquidity risk are priced and it imposes the model constraint of a single risk price, then : $\beta^{Net}_{i,t}=\beta^1_{i,t}+\beta^2_{i,t}-\beta^3_{i,t}-\beta^4_{i,t}$. Equation 3 estimates an unconstrained model, allowing different liquidity premiums as in Acharya and Pedersen (2005). This equation is likely to suffer from the multicollinearity of the betas. Equations 4, 5 and 6 aim to overcome this issue by estimating the three liquidity risk premiums separately. Finally equation 7 isolates the effect of liquidity risk ($\beta^{Liq}_{i,t}=\beta^2_{i,t}-\beta^3_{i,t}-\beta^4_{i,t}$) from the effect of market risk ($\beta^1_{i,t}$). All equations are estimated in a cross-sectional framework where $i=1:25$ and $t=1:120$. The constant α is added to evaluate possible pricing errors. The risk free rate is proxy by the 2 years US treasury bond.

Cross-sectional regressions are estimated similarly to that in the second step of the Fama-MacBeth procedure. That is to say that, for each month, we run 25 regressions, the coefficients are then averaged and standard errors are Newey-West-adjusted with two lags.

[14] Excess returns are adjusted by the liquidity level premium ($-k_{i,t}E(c_{i,t})$), thus it does not correspond exactly to the CAPM model

Pseudo R-Squared (R2) and pseudo adjusted R-Squared (Adj-R2) are computed as, respectively, average of the individuals R-Squared and adjusted R-Squared.

The original L-CAPM has been tested with a constant holding period (Acharya and Pedersen, 2005). However Hagstromer et al. (2013) point out that the holding period parameter (k) is time-varying and should be included as such (k_t) to avoid over- or under-estimation of illiquidity costs. We follow their recommendation but improve it by allowing this parameter to differ *across* portfolios ($k_{i,t}$). Indeed, it is known that illiquid stocks tend to be held by long term investors and liquid stocks by short term investors (Amihud and Mendelson, 1986). Then, assuming that the typical holding period of investors is constant across portfolios probably leads to overestimation of the liquidity level premium for low liquid stocks and underestimation for high liquid stocks. We compute this parameter for each portfolio as the inverse of the turnover ratio (appropriately re-scaled to a monthly frequency). Unreported results show a clear negative relationship between illiquidity and parameter k. This was expected because the turnover ratio of low liquid stocks is lower than that of high liquid stocks.

Cross section regressions are estimated for the four models summarised in Table 1 and presented in section 3.3. Table 8 displays the average adjusted R-Square per model. As a main result we found that all models have similar R2 with the "simplest" model specification (model I), is the one which best explains the cross-section of returns (average adjusted R-square: 37.24 %).

Table 8: Adjusted R2 per model. This table reports the adjusted R-square of the different cross sections regressions for the four models implemented.

Models	I	II	III	IV
Equation 1	34,827%	34,747%	33,973%	33,973%
Equation 2	34,916%	35,011%	30,277%	30,276%
Equation 3	41,689%	40,216%	42,656%	42,655%
Equation 4	40,193%	37,685%	39,073%	39,073%
Equation 5	36,412%	37,063%	35,870%	35,871%
Equation 6	35,989%	35,604%	35,286%	35,286%
Equation 7	36,683%	37,141%	38,388%	38,388%
Average	37,244%	36,781%	36,503%	36,503%

In consequence, and also for ease of exposure, only estimates of risk premia corresponding to model I are presented[15]. Tables 9 and 10 display cross sectionals results for, respectively, the conditional and unconditional version of the L-CAPM estimated via time-varying betas obtained using model I.

As important result we find that liquidity risk is priced, indeed equations 2-7 (Eq2-Eq7) better explains the cross sections of returns than equation 1. From table 9 we can also see that the coefficient $\lambda_{Liq} = 0.203$ of equation 7 which isolates the liquidity risk is found to be positive and significant. However given the aforementioned high degree of correlation between

[15] Cross-sections results of models II, III and IV are available from the authors on request.

β_1 and β_{Liq}, this result should be interpreted with caution. Equations Eq3 to Eq6 suffer from the same multicollinearity problem. Nevertheless, the liquidity risk attached to β_2,the portfolio liquidity sensitivity to market illiquidity seems to be the predominant liquidity risk as it is the only risk with significant coefficient in Eq3 and Eq4 (table 9).

As expected, the standard market beta (β_1) is also priced in nearly all equations as indicated by the significance of the λ_1 coefficient. Importantly, in term of goodness-of-fit, we observe that for all pricing equations, the constant term is not significantly different from zero, indicating that models have zero average pricing error and that the theoretical model fits well to the data. By allowing different liquidity premiums (Eq3), we observe an overall increase in the adjusted r-square to 41.689 %, an observation which tends to reject the model constraint of a single liquidity premium. The control variables SIZE and PBR are found to be significant determinants of returns, with, as usual, the SIZE variable having a negative impact on returns and the PBR a positive effect.

Table 9: Conditional L-CAPM - Fama-MacBeth regressions. This table reports the estimated coefficients from cross-sectional regressions of the liquidity-adjusted CAPM for 25 equally-weighted portfolios using monthly data during 2006-2015. It reports the coefficient average together with corresponding Fama MacBeth (adjusted)t statistics. The factor loadings have been estimated using model I (DCC) as presented in section 3.2. $R2_s$ ($ADJR2$) refer to the (adjusted) pseudo coefficient of determination.

Coeff.	[Eq1]	[Eq2]	[Eq3]	[Eq4]	[Eq5]	[Eq6]	[Eq7]
α	.0106 (.0081)	.0103 (.0081)	-.0061 (.0076)	-.0083 (.0079)	.0098 (.0074)	.0057 (.0070)	-.0005 (.0064)
λ_1	.0063** (.0035)		.0038 (.0040)	.0079** (.0034)	.0021 (.0038)	.0069** (.0037)	.0067** (.0034)
λ_2			2.4006*** (.8661)	2.0757*** (.8043)			
λ_3			.0316 (.3900)		-0.2601 (.3882)		
λ_4			.1893 (.1512)			-.1366 (.1223)	
λ_{Net}		.0064** (.0035)					
λ_{Liq}							.2032** (.1099)
α_s	-.0031*** (.0007)	-.0031*** (.0007)	-.0011** (.0006)	-.0014** (.0006)	-.0030*** (.0006)	-.0026*** (.0006)	-.0021*** (.0005)
α_p	.0066*** (.0016)	.0066*** (.0016)	.0056*** (.0014)	.0067*** (.0015)	.0065*** (.0015)	.0066*** (.0015)	.0066*** (.0014)
$ADJR2$	.34827	.34916	.41689	.40193	.36412	.35989	.36683
$R2_s$	.42974	.43052	.56266	.50161	.4701	.46657	.47236

Table 10: Unconditional L-CAPM - Fama-MacBeth regressions. This table reports the estimation results for the Fama-MacBeth regressions of the *Unconditional* liquidity-adjusted CAPM See legend in Table 9 for more details.

Coeff.	1	2	3	4	5	6	7
α	.0091 (.0081)	.0091 (.0082)	-.0314** (.0140)	-.0226** (.0126)	.0100 (.0083)	-.0109 (.0106)	-.0176 (.0116)
λ_1	.0012 (.0038)		.0161*** (.0054)	.0136*** (.0051)	.0015 (.0039)	.0103** (.0044)	.0124*** (.0046)
λ_2			6.8837** (3.6136)	8.5508*** (3.0305)			
λ_3			-2.2510** (1.1184)		.4114 (1.0236)		
λ_4			-.3898 (0.2515)			-.7566*** (.2306)	
λ_{Net}		.0013 (.0039)					
λ_{Net}		.0017 (.0038)					
λ_{Liq}							.8297*** (.2446)
α_s	-.0021*** (.0006)	-.0021*** (.0006)	-.0001 (.0007)	-.0002 (.0006)	-.0021*** (.0007)	-.0007 (.0006)	-.0004 (.0006)
α_p	.0066*** (.0017)	.0066*** (.0017)	.0071*** (.0018)	.0073*** (.0019)	.0068*** (.0018)	.0061*** (.0015)	.0061*** (.0016)
$ADJR2$	.33125	.3311	.43423	.39642	.33578	.36148	.36808
$R2_s$	.41484	.41471	.57567	.49702	.44648	.4679	.4734

4.4. Decomposition of premiums

In the following paragraph we analyse and decompose the liquidity premium as in Hagstromer et al. (2013). We follow the model constraint of a single risk price and base our analysis on estimates with model I.

The total liquidity premium consists of:

- the illiquidity level compensation ($LLP_{i,t}$) computed as $LLP_{i,t} = \lfloor k_{i,t} E(c_{i,t}) \rfloor \times 12$,
- the total liquidity risk premium ($TLRP_{i,t}$): $TLRP_{i,t} = \lambda_{Net} \times \beta^{Net}$ (from Eq 2).

The latter premium can itself be decomposed into three different premiums referring to the three liquidity risks. Thus the annual total liquidity risk premium ($TLRP_{i,t}$) is given by $TLRP_{i,t} = LRP1_{i,t} + LRP2_{i,t} + LRP3_{i,t}$ where $LRP1, LRP2$ and $LRP3$ are respectively the liquidity risk premium referring to β^2, β^3 and β^4 (β^1 being the sensitivity to the market). According to the theoretical model, and assuming a single price of risk (λ_{net}), the annual total liquidity risk premium is computed as:

$$TLRP_{i,t} == \beta_{i,t}^{2} * \lambda_{net} * 12 + \beta_{i,t}^{3} * \lambda_{net} * 12 + \beta_{i,t}^{4} * \lambda_{net} * 12$$

where the factor 12 is used to annualise the monthly premiums. The λ_{net} coefficient is obtained at the previous cross section stage ($\lambda_{net} = 0.006$ in Eq2, Table 9). Note that even though the price of risk is constant (λ_{net}), risk premiums obtained are time varying due to the time varying behaviour of the betas. Finally, the total premium (TP) consists of the three liquidity risks premiums ($LRP1 + LRP2 + LRP3$), the standard market risk premium (RP) and the liquidity level premium (LLP): $TP = LLP + RP + LRP1 + LRP2 + LRP3$. Tables 11 and 12 display the liquidity risk premium decomposition for each portfolio.

The annual liquidity level premium (LLP) has a mean of 0.455% and is found to increase with illiquidity (from 0.136 % for portfolio 1 to 1.410 % for portfolio 25). This increasing relationship can be explained by the " the clientele effect " proposed by Amihud and Mendelson (1986): agents with a longer expected holding period can depreciate the trading costs over a longer period than short terms investors and benefit from a higher adjusted liquidity return. These values have the same order of magnitude as the results of Hagstromer et al. (2013), who found a liquidity level premium within the range 0.098 % to 1.38 % . The market risk premium is found to be by far the most important risk factor with an average annual value of 7.625 %. In addition, it is negatively correlated with illiquidity implying that liquid stocks are more subject to market risk than illiquid stocks. This value is much higher than the average total liquidity risk premium (0.175%) denoting that the main concern of investors is market risk not liquidity risk.

We also found that the total liquidity risk premium (TLRP) is positively related to illiquidity, portfolio 1 has an average TLRP value of 0.112 % and portfolio 25, 0.303 % . At first sight, this seems to show that illiquid portfolios have a greater liquidity risk than liquid portfolios. However, we found the opposite regarding one particular liquidity risk: we observe a positive correlation between liquidity and LRP2. LRP2 has an average value of 0.074 % for portfolio 1 (the most liquid portfolio) and 0.047 % for portfolio 25. As far as we read, it is the first time that the premium of a liquidity risk is found to *increase* with liquidity. The price reward to bear the risk to obtain low returns during a liquidity crisis period is higher for liquid stocks than for illiquid stocks because liquid stocks are more exposed to this risk. This shows that agents investing in liquid portfolios are highly concerned with their return sensibility to the liquidity of the market (β_3).

The previous analysis focused on raw premiums, in table 12 we evaluate the *relative* importance of each liquidity premium for each portfolio. Indeed, it also seems relevant to study the importance of each liquidity premium with respect to the total liquidity premium since a higher value of a premium in absolute terms does not by itself indicate agents' preferences [16]. We therefore compute for each liquidity risk, its proportion with respect to the total liquidity premium. As a general observation, we found that the portion of the total liquidity premium due to the illiquidity level premium is far more important than the portion due to liquidity risk, with values of 51.3 % for portfolio 1 and 81.21 % for portfolio 25. This portion is similar to that found by Hagstromer et al. (2013) which is approximately 75 %. This indicates that liquidity risk

[16] It may only means that the exposure (beta) to a specific risk is higher.

plays the minor role in explaining returns and that the liquidity level plays the major role. This proportion increases with illiquidity, indicating that long term investors are more interested in having an important return compensation to hold an illiquid portfolio (LLP) than in obtaining compensation for the associated liquidity risk. Regarding liquidity risk, only the proportion of LRP2 over the total of liquidity premium displays a clear negative correlation with illiquidity. This confirms our previous finding that agents who favour liquid portfolios (short terms investors) are more concerned with the risk to get low returns when the liquidity of the market decreases (β_3) than agents who invest in illiquid stocks.

By computing the difference between premiums of portfolios 1 and 25, and by using Acharya and Pedersen (2005) methodology, we can obtain an approximation for the annualised return difference resulting from the difference in illiquidity. Thus the annualised return difference due to β^3, the sensitivity of the portfolio's returns to market illiquidity, is computed as the time-series average : $\Delta Ret(\beta^3) = E(LRP2_{25t} - RP2_{1t})$. Using the Newey-West-adjusted standard error of λ_{Net} and the betas, we also compute the 95% confidence interval. The results are presented in Table 13. The annualised return difference due to β^2, β^3 and β^4 are respectively 0.071 %, -0.027 % and 0.147 %, implying an average total annualised effect on returns of the liquidity risk of 0.191 %. The annualized return difference due to LLP is found to be 1.274 % and the total return difference due to liquidity risk and illiquidity level sum up to 1.465 %. This shows that the effect of liquidity risk on returns is marginal compared with the effect of the level of illiquidity. In total the impact of illiquidity on return (1.465%) is significantly lower than that found by Acharya and Pedersen (2005) (4.6 %) and is close to the value found by Hagstromer et al. (2013) ([1.74%: 2.08 %]). In line with the previously documented positive relation between liquidity and LRP2, we found that the risk related to the portfolio return sensitivity to market illiquidity (LRP2) has a (slightly) negative effect ($\Delta_r^{\beta^3} = -0.027\%$) on returns. This is due to the fact that the LRP2 received by short term investors (liquid portfolios) is higher than that received by long term investors, which creates an overall negative effect on returns.

The highest priced liquidity risk is LRP3, indicating that agents are willing to accept a high discount for portfolios which are liquid in periods of low returns. This was also found by Hagstromer et al. (2013) and may be explained by the theoretical model provided by Wagner (2011). He argues that during crisis period investors face an important liquidation risk and this creates incentives to allocate their portfolio to liquid assets. Finally, the market risk is found to have a stronger impact on returns than illiquidity, the annualised return stemming from it is $\Delta_r^{\beta^1} = -4{,}483\%$ (95% confidence interval : [-4.675 %;-4.291 %]). The negative sign is due to the fact that the market risk premium is higher for liquid stocks than for illiquid stocks.

Table 11: Price decomposition. This table reports the average annual price of the different risks factors in percentage. MRP is the market risk price (β_1), and LLP the price of the liquidity level compensation.LRP1, LRP2 and LRP3 are the price of the three liquidity risk and LRPT refers to

the total liquidity risk premium (TLRP = LRP1 + LRP2 +LRP3).

Port	MRP	LLP	E(TLRP)	σ^2 (TLRP)	E(LRP1)	σ^2 (LRP1)	E(LRP2)	σ^2 (LRP2)	E(LRP3)	σ^2 (LRP3)
1	7.816	0.136	.112	.00049	.018	.00011	.074	.00029	.02	.00016
2	8.929	0.179	.113	.000566	.022	.000126	.083	.000358	.009	.00018
3	9.166	0.189	.133	.000684	.022	.000128	.084	.000392	.027	.00025
4	9.399	0.193	.127	.00064	.022	.000128	.083	.000382	.022	.00025
5	9.833	0.217	.13	.000733	.025	.000149	.084	.000435	.02	.00029
6	9.805	0.215	.148	.000744	.024	.000144	.088	.000451	.036	.00029
7	9.794	0.23	.158	.000747	.025	.000156	.086	.000408	.047	.00028
8	9.785	0.235	.162	.000783	.026	.000157	.086	.000415	.05	.00033
9	10.38	0.236	.152	.000771	.026	.000163	.083	.000411	.043	.0003
10	10.15	0.254	.136	.000607	.027	.000163	.078	.000338	.032	.00019
11	9.847	0.265	.169	.000689	.028	.000176	.077	.000289	.063	.00028
12	9.83	0.259	.154	.000552	.028	.000165	.078	.000239	.048	.00020
13	9.358	0.312	.16	.000635	.029	.000183	.07	.000247	.061	.00026
14	8.885	0.317	.159	.000717	.025	.000163	.068	.0004	.065	.00033
15	8.401	0.375	.178	.00092	.027	.000176	.075	.000414	.076	.0005
16	7.656	0.387	.162	.000596	.025	.000159	.074	.000279	.063	.00032
17	6.981	0.478	.157	.000649	.022	.00013	.074	.000338	.061	.00033
18	6.47	0.612	.132	.000616	.02	.000121	.064	.000301	.049	.00033
19	5.785	0.631	.209	.000876	.04	.000232	.056	.000272	.113	.00049
20	4.749	0.697	.239	.000944	.056	.000309	.051	.000224	.131	.00054
21	4.334	0.783	.24	.000838	.046	.000269	.052	.000214	.142	.00045
22	3.507	0.808	.262	.00101	.052	.000259	.051	.000243	.159	.00070
23	3.109	1.012	.188	.000763	.04	.000205	.046	.000255	.102	.00059
24	3.312	0.95	.303	.00111	.055	.000302	.04	.00023	.208	.00067
25	3.333	1.41	.303	.00169	.089	.000755	.047	.000466	.168	.00077
Average	7.625	0.455	.175	.000775	.033	.000201	.07	.000331	.073	.00037

Table 12: Liquidity Premium decomposition. This table reports the proportion of each liquidity premium with respect to the total liquidity premium. LLP, LRP1, LRP2 and LRP3 refer, repectively, to the Liquidity Level Premium and the three Liquidity Risks premiums related to β^2, β^3 and β^4.

Port	LLP	LRP1	LRP2	LRP3
1	51,30%	7,78%	32,26%	8,66%
2	58,13%	8,21%	31,11%	2,55%
3	56,47%	7,27%	27,96%	8,29%
4	58,48%	7,29%	27,31%	6,92%
5	60,64%	7,95%	26,13%	5,28%
6	57,69%	7,04%	24,98%	10,29%
7	57,76%	6,79%	22,89%	12,56%
8	58,07%	6,85%	22,26%	12,82%
9	60,38%	7,02%	21,72%	10,87%
10	63,74%	7,03%	20,73%	8,50%
11	60,17%	6,61%	18,28%	14,94%
12	62,37%	6,65%	19,29%	11,69%
13	64,34%	6,20%	15,88%	13,57%
14	66,72%	5,23%	14,33%	13,72%
15	68,32%	4,73%	13,71%	13,24%
16	70,27%	4,47%	13,81%	11,45%
17	74,81%	3,48%	11,97%	9,75%
18	81,50%	2,70%	9,09%	6,71%
19	74,86%	4,70%	6,71%	13,73%
20	74,20%	5,99%	5,56%	14,26%
21	75,63%	4,60%	5,28%	14,49%
22	74,80%	5,01%	4,90%	15,28%
23	82,64%	3,61%	4,26%	9,49%
24	75,30%	4,37%	3,19%	17,15%
25	81,21%	5,18%	2,81%	10,79%

Table 13: Annual liquidity premiums effects on returns. Annualized estimated time-series averages of liquidity premiums 95% Confidence interval. MRP refers to the standard market risk premium.

	Annualized return	Lower 95% bound	Upper 95% bound
TLP	1,465%	1,456%	1,473%
LLP	1,274%		
$\Delta_r(TLRP)$	0,191%	0,183%	0,199%
$\Delta_r^{\beta^2}(LRP1)$	0,071%	0,068%	0,074%
$\Delta_r^{\beta^3}(LRP2)$	-0,027%	-0,026%	-0,028%
$\Delta_r^{\beta^4}(LRP3)$	0,147%	0,141%	0,154%
$\Delta_r^{\beta^1}(MRP)$	-4,483%	-4,291%	-4,675%

5. Conclusion

To conclude, we propose a new methodology to test the conditional L-CAPM model based on modelling illiquidity using Unobserved Components models. Our results can help portfolio managers to better allocate their assets by evaluating properly the liquidity premium. Our results confirm some of the previous empirical founding such as the marginal effect of liquidity risk on returns compare to the effect of the liquidity level premium. However, one important observation is in contraction with previous founding, liquidity risk and illiquidity level are not found to be always positively correlated.

Regarding the bid-ask spread our key results are 1) bid-ask spread exhibits strong periodicity; 2) the relative importance of temporary shocks over permanent shocks varies according to the liquidity level of the portfolio; 3) bid-ask spreads are highly impacted by temporary shocks; 4) the bid-ask spread variance process is highly persistent; 5) there exists an overall increasing trend in liquidity during the sample period (2006-2015) and that volatility clustering exists in bid-ask spread series.

As far as we read, the previous key result has never been jointly considered when testing the L-CAPM model. By taking into account all these features, by limiting the look-ahead bias encountered in previous studies and by appropriately considering the holding period as varying across portfolios, we believe that our results are reliable.

As main empirical result, we found that 1) the effect of liquidity risk on returns is marginal compared with the effect of the liquidity level premium; 2) the liquidity risk related to the covariance between portfolio illiquidity and market return (LRP3) is the most important risk; 3) liquidity risk is not always positively correlated with illiquidity, it is the case for LRP2 - this may lead to new risk hedging strategy; 4) liquidity risk is time-varying and priced, and 5) liquidity risks are co-moving.

As possible future research it could be worthwhile extending the UC modelling framework to a multivariate framework, to characterise the documented multicollinearity of the betas more precisely.

1 Appendix: Derivation of the Kalman Filter

In this appendix we derive the equations of the Kalman filter that are used to estimate the Unobserved Components models. We focus on a simplest state space representation in which we assume system matrices to be known and Gaussian Distributed, it is the case for the LLT model. The transition equation is given by:

$$\alpha_{t+1} = T_t \alpha_t + R_t \zeta_t$$

and the observation equation is: $y_t = Z_t \alpha_t + \varepsilon_t$

with:

$$\varepsilon_t : N(0, H_t) \rightarrow H_t = E[\varepsilon_t \varepsilon_t^T]$$
$$\zeta_t : N(0, Q_t) \rightarrow Q_t = E[\zeta_t \zeta_t^T]$$
$$\alpha_1 : N(a_1, P_1)$$
$$t = 1, \ldots, n$$

Kalman Filter is used to predict the mean ($E(\alpha_{t+1} | Y_t)$) and variance ($Var(\alpha_{t+1} | Y_t)$) of the unobserved state vector α_t.

Let Y_t denotes observations up to time t ($\{y_1, y_2, \ldots, y_t\}$), a_{t+1} the state prediction and P_{t+1} the variance of the estimate:

$$a_{t+1} = E(\alpha_{t+1} | Y_t)$$
$$P_{t+1} = Var(\alpha_{t+1} | Y_t)$$

The prediction error (v_t) is given by the difference between the observation (y_t) and the predicted value ($E(y_t | Y_{t-1})$):

$$v_t = y_t - E(y_t | Y_{t-1}) = y_t - Z_t E(\alpha_t | Y_{t-1}) = y_t - E(Z_t \alpha_t + \varepsilon_t | Y_{t-1}) = y_t - Z_t a_t$$

where we used the fact that $E(\varepsilon_t | Y_{t-1}) = 0$ to simplify the third equality. By replacing y_t with $y_t = Z_t \alpha_t + \varepsilon_t$ we obtain:

$$v_t = y_t - Z_t a_t = [Z_t \alpha_t + \varepsilon_t] - Z_t a_t = Z_t(\alpha_t - a_t) + \varepsilon_t$$

Moreover since by definition $E(\alpha_t - a_t) = 0$, then $E(v_t) = 0$.

Thus the prediction error has a zero mean. It follows that the prediction error variance denoted $F_t = Var(v_t)$ is [17]

$$F_t = Var(v_t) = Var\left[Z_t(\alpha_t - a_t) + \varepsilon_t\right]$$

[17] note that 1) ε_t is independent of α_t and a_t then all cross products involving these terms disappear; 2) P_t is defined above and that the superscript ^{T} denotes the transpose operator. Also recall that the transpose of a product of matrices equals the product of their transposes in reverse order: $(AB)^T = B^T A^T$

$$= E\left[(Z_t(\alpha_t - a_t) + \varepsilon_t)((\alpha_t - a_t)^T Z_t^T + \varepsilon_t^T)\right]$$
$$= Z_t E(\alpha_t - a_t)(\alpha_t - a_t)^T Z_t^T + E(\varepsilon_t \varepsilon_t^T)$$
$$= Z_t P_t Z_t^T + H_t$$

To continue, we need to use the following lemma 0.1 that is originated from the multivariate normal regressions theory:

Lemma 0.1 *if x; y and z are jointly normally distributed vectors with* $E(z) = 0$ *and* $\Sigma_{yz} = 0$ *(where Σ is the covariance matrix). Then* $E(x \mid y, z) = E(x \mid y) + \Sigma_{xz}\Sigma_{zz}^{-1} z$ and variance matrix: $Var(x \mid y, z) = Var(x \mid y) - \Sigma_{xz}\Sigma_{zz}^{-1}\Sigma_{xz}^T$

By taking $x = \alpha_t$; $y = Y_t$ and $z = v_t$, the precedent lemma can be employed because all vectors are Gaussian distributed and because $\Sigma_{Y_t, v_t} = 0$. Let's first compute the covariance matrix Σ_{α_t, v_t} :

$$\Sigma_{\alpha_t, v_t} = E[(\alpha_t - a_t)v_t^T] = E[(\alpha_t - a_t)(Z_t\alpha_t + \varepsilon_t - Z_t a_t)^T] = E[(\alpha_t - a_t)(\alpha_t - a_t)^T]Z_t^T = P_t Z_t^T$$

By applying lemma 0.1:

$$E(\alpha_{t+1} \mid Y_t) = E(\alpha_{t+1} \mid Y_{t-1}, v_t)$$
$$= E(\alpha_{t+1} \mid Y_{t-1}) + \Sigma_{\alpha_{t+1}, v_t}\Sigma_{v_t, v_t}^{-1} v_t = E(\alpha_{t+1} \mid Y_{t-1}) + \Sigma_{\alpha_{t+1}, v_t} F_t^{-1} v_t = T_t a_t + K_t v_t$$

where we have replaced the term $\Sigma_{\alpha_{t+1}, v_t} F_t^{-1}$ by a new term K_t that is called the Kalman Gain[18]. Equation shows that best prediction of the next state mean is given by a linear combination of the current estimate of the state mean a_t and a second term which depends of the prediction error variance v_t corrected by the Kalman Gain (K_t).

The Kalman gain can itself be decomposed as [19]:

$$K_t = \Sigma_{\alpha_{t+1}, v_t} F_t^{-1} = E[\alpha_{t+1} v_t^T \mid Y_{t-1}] \times F_t^{-1}$$
$$= E[(T_t\alpha_t + R_t\zeta_t)v_t^T \mid Y_{t-1}] \times F_t^{-1} = (E[T_t\alpha_t v_t^T] + E[R_t\zeta_t v_t^T]) \times F_t^{-1} = T_t E[\alpha_t v_t^T] \times F_t^{-1} = T_t P_t Z_t^T F_t^{-1}$$

Similarly by applying the lemma 0.1 to the variance of the estimate [20]:

$$Var(\alpha_{t+1} \mid Y_{t-1}, v_t) = Var(\alpha_{t+1} \mid Y_{t-1}) - \Sigma_{\alpha_{t+1} v_t}\Sigma_{v_t v_t}^{-1}\Sigma_{\alpha_{t+1} v_t}^T$$

---

[18] We remind that that $F_t = \Sigma_{v_t, v_t}$

[19] We remind that v_t and ζ_t are uncorrelated in the fourth equality.

[20] recall also that $\Sigma_{\alpha_{t+1}, v_t} = T_t P_t Z_t^T$ as demonstrated in ??), and then its transpose is given by $\Sigma_{\alpha_{t+1}, v_t}^T = Z_t P_t^T T_t^T$

$$=Var(T_t\alpha_t + R_t\zeta_t \mid Y_{t-1}) - T_tP_tZ_t^T F_t^{-1} Z_tP_t^T T_t^T$$
$$=T_tVar(\alpha_t \mid Y_{t-1})T_t^T + R_tQR_t^T - T_tP_tZ_t^T F_t^{-1} Z_tP_t^T T_t^T$$
$$=T_tP_tT_t^T + R_tQR_t^T - T_tP_tZ_t^T F_t^{-1} Z_tP_t^T T_t^T$$
$$=T_tP_tT_t^T + R_tQR_t^T - K_tF_tK_t^T$$

Then we develop the so called "update equations", which are defined by $a_{t|t} = E(\alpha_t \mid Y_t)$ and $P_{t|t} = Var(\alpha_t \mid Y_t)$. By applying lemma 0.1 :

$$a_{t|t} = E(\alpha_t \mid Y_t) = E(\alpha_t \mid Y_{t-1}, v_t) = E(\alpha_t \mid Y_{t-1}) + \Sigma_{\alpha_t, v_t} \Sigma_{v_t, v_t}^{-1} v_t = a_t + P_tZ_t' F_t^{-1} v_t$$

And $P_{t|t} = Var(\alpha_t \mid Y_{t-1}, v_t) = Var(\alpha_t \mid Y_{t-1}) - \Sigma_{\alpha_t v_t} \Sigma_{v_t v_t}^{-1} \Sigma_{\alpha_t v_t}' = P_t - P_tZ_t' F_t^{-1} Z_tP_t$

The Kalman Filter consists of seven filtering equations ("Kalman Equations") that we have derived previously:

$$v_t = y_t - Z_ta_t$$
$$F_t = Z_tP_tZ_t' + H_t$$
$$K_t = T_tP_tZ_t'F_t^{-1}$$
$$a_{t+1} = T_ta_t + K_tv_t$$
$$P_{t+1} = T_tP_tT_t' + R_tQR_t' - K_tF_tK_t'$$
$$a_{t|t} = a_t + P_tZ_t' F_t^{-1} v_t$$
$$P_{t|t} = P_t - P_tZ_t'F_t^{-1}Z_tP_t$$

for $t = 1, \ldots, n.$. It is an algorithm that is apply recursively, first the prediction of the state is given by K3 and the prediction of its variance by K4. Then at every new observation that becomes available we compute the error in prediction (K0), update the system using the equation K5 and K6. This permits to obtain a new Kalman Gain (K2) and a new prediction error variance (K1). Finally a new prediction for the state and its variance can be made and the algorithm continues up to the end of the sample. The log-likelihood is computed as a by-product of the Kalman Filter by using F_t and v_t.

We have demonstrated the derivation of the conditional mean vector and conditional variance matrix of the state vector α_t given the observations $y_1, \ldots, y_t$ (filtering). Similarly, Kalman equations can be derived for the conditional mean vector and variance matrix given, respectively, $y_1, \ldots, y_{t-1}$ (our case) and $y_1, \ldots, y_n$ to obtain the prediction and smoothing equations(see [Harvey, 1994, Durbin and Koopman, 2012]).

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