Structural Pricing of CoCos and Deposit Insurance with Regime Switching and Jumps *

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Abstract This article presents a structural model with jumps and regime switching features that is specifically dedicated to the pricing of bank contingent convertible debt (CoCos) and deposit insurance. This model assumes that the assets of a bank evolve as a geometric regime switching double exponential jump diffusion and that debt profiles are exponentially decreasing with respect to maturity. The paper starts by giving a general presentation of the jumps and regime switching framework, where an emphasis is put on the definition of an Esscher transform applicable to regime switching double exponential jump diffusions. The following developments concentrate on the definition and implementation of a matrix Wiener-Hopf factorization associated with the latter processes. Then, valuation formulas for the bank equity, debt, deposits, CoCos and deposit insurance are obtained. An illustration concludes the paper and addresses the respective impacts of jumps and regime switching on the viability of the bank.

Keywords: Matrix Wiener-Hopf factorization. Esscher transform. Jump-diffusion. Regime switching. COCOs. Deposit insurance. Structural model. Fluid embedding. Markov chain.

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1 Introduction

One of the first structural models for corporate debt pricing dates back to Merton (1974). Black and Cox (1976) extend this model by allowing default to occur prior to debt maturity. Then, Longstaff and Schwartz (1995) extend the Black and Cox model by introducing interest rate risk. Leland (1994a), Leland (1994b) and Leland and Toft (1996) characterize the optimal capital structure of a firm as a trade-off between bankruptcy costs and tax shield. These models all use geometric Brownian motion to represent the dynamics of the firm asset value. Introducing jumps into asset dynamics solves the problem of credit spreads becoming zero when the bond maturity decreases to zero: Hilberink and Rogers (2002) incorporate negative jumps into the dynamics of the firm asset value, while Chen and Kou (2009) propose a structural model with two-sided jumps based on the double exponential jump diffusion process. Because economic environments substantially change over long periods, it is useful to incorporate a regime switching behavior into structural models. Hainaut, Shen, and Zeng (2016) use a regime switching Brownion motion to model the dynamics of EBIT in a capital structure model.

In this paper, we introduce a structural model that combines the jump and regime switching features, concentrates on the dynamics of assets rather than on those of EBIT, and postulates an exponentially decreasing profile of debt with respect to maturity. We apply this model to a bank and price CoCos and deposit insurance. The capital structure of this bank is assumed to include bonds, contingent convertible bonds, equity, deposits and deposits insurance. The bankruptcy is assumed to happen after the conversion of CoCos and both the conversion time of CoCos and the bankruptcy time are assumed to occurr when the ratio of debt face value to asset value crosses an exogenous level.

The pricing of CoCos and deposit insurance should be conducted under the risk neutral measure. Le Courtois and Quittard-Pinon (2006) show that the double exponential jump diffusion process keeps the same structure when using the Esscher transform measure. Elliott, Chan, and Siu (2005) construct the Esscher transform for the regime switching geometric Brownian motion and Hainaut, Shen, and Zeng (2016) illustrate this latter result. Elliott, Siu, Chan, and Lau (2007) construct the Esscher transform for the generalized regime switching jump diffusion model where the interest rate, drift rate and volatility are regime switching.

This paper defines in section 2 a general Esscher transform that preserves the structure of the regime switching double exponential jump diffusion process where the interest rate, drift rate, volatility, but also jump intensity and jump distribution are regime switching. Section 3 shows how it is possible, using results of Jiang and Pistorius (2008), to express and compute the matrix Wiener-Hopf factors for a process that switches between two jump diffusions with double exponential jump distributions. Building on the general Esscher transform and on the matrix Wiener-Hopf factorization, we provide in section 4 closed-form formulas for the value of the bank's equity, debt, deposits, CoCos and deposit insurance. Finally, we provide in section 5 a numerical experiment where we study the influence of the regime switching behavior and of jump risk on the conversion and default probabilities and on the pricing of CoCos and deposit insurance.

2 The Regime Switching Jump Diffusion Structural Model

We define a continuous time Markov chain process $J = \{J_t; t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ with a finite state space $E^0 = \{e_1, e_2, ..., e_n\}$, where $e_i = (0, ..., 1, ..., 0) \in \mathbb{R}^n$. These states represent the states of the economy. The value of the bank assets is assumed to follow an exponential regime switching jump diffusion process under the real-world probability measure P:

$$V_t = V_0 e^{X_t},$$

where V_0 is the initial bank asset value and X is a regime switching jump diffusion process:

$$X_t \quad = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \mathrm{d}N_s,$$

where W is a standard Brownian motion, $\mu_t = \langle \hat{\mu}, J_t \rangle$, $\sigma_t = \langle \hat{\sigma}, J_t \rangle$, $N_t = \langle \hat{N}, J_t \rangle$, and where $\langle ., . \rangle$ denotes the inner product, $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_n)$, $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, ..., \hat{\sigma}_n)$ and $\hat{N} = (\hat{N}_1, \hat{N}_2, ..., \hat{N}_n)$. For each state $e_i \in E$, $\hat{\mu}_i \in \mathbb{R}$, $\hat{\sigma}_i \geq 0$, $\hat{N}_i = \{\hat{N}_i(t); t \geq 0\}$ is a compound Poisson process with rate $\hat{\lambda}_i$ and the jumps size is modeled with an asymmetric double exponential distribution of density function:

$$f_i(y) = p_i \hat{\eta}_{1i} e^{-\hat{\eta}_{1i} y} I_{\{y \geq 0\}} + q_i \hat{\eta}_{2i} e^{\hat{\eta}_{2i} y} I_{\{y < 0\}},$$

where $\hat{\eta}_{1i} > 1$, $\hat{\eta}_{2i} > 0$, $p_i \geq 0$, $q_i \geq 0$, $p_i + q_i = 1$. The stochastic processes $\{W_t; t \geq 0\}$ and $\{\hat{N}_i(t); t \geq 0\}$ are independent. Denote X at state e_j as X^j and the filtration generated by J as $\mathcal{G}$. $\mathcal{G}$ is augmented as $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$ where $\mathcal{H}$ is the filtration generated by the $\{X^j; j = 1, 2, ..., n\}$.

Suppose that the generator matrix of J is

$$Q = \{q_{ij}\}_{1 \leq i,j \leq n},$$

where $q_{ii} = -\sum_{i \neq j} q_{ij}$. The $(i,j)^{\text{th}}$ element q_{ij} represents the transition rate at which the process J jumps from state e_i to state e_j. Then, the transition probabilities matrix is

$$P(s,t) = e^{Q(t-s)} \quad \forall s \leq t$$

and the $(i,j)^{\text{th}}$ element $p_{i,j}(s,t)$ of this matrix is the probability of switching from state e_i at time s to state e_j at time t. Denote the moment generating function of X_t as $M_t(u)$, an $N \times N$ matrix with $(i,j)^{\text{th}}$ element equal to $E(e^{uX_t}; J_t = e_j | J_0 = e_i)$. Then, $M_t(u) = e^{tZ(u)}$, where

$$Z(u) = Q + \operatorname{diag}\{\varphi_j(u)\},$$

and $\varphi_j(u)$ is the Laplace exponent of X^j under state e_j defined as follows:

$$\varphi_j(u) = \hat{\mu}_j u + \frac{1}{2}\hat{\sigma}_j^2 u^2 + \hat{\lambda}_j \left(\frac{p_j \hat{\eta}_{1j}}{\hat{\eta}_{1j} - u} + \frac{q_j \hat{\eta}_{2j}}{\hat{\eta}_{2j} + u} - 1 \right).$$

Now, we define a non-negative $\mathcal{F}-$adapted stochastic process θ as $\{\theta_t = \langle J_t, \hat{\theta} \rangle; t \geq 0\}$ where $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_n)$. Then, the regime switching Esscher measure $\tilde{P}$ is defined as follows:

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}|_{\mathcal{F}_t} = \frac{e^{\int_0^t \theta_s \mathrm{d}X_s}}{E_P\left(e^{\int_0^t \theta_s \mathrm{d}X_s} | \mathcal{G}_t\right)} = e^{\int_0^t \theta_s \sigma_s \mathrm{d}W_s - \frac{1}{2}\int_0^t {\theta_s}^2 {\sigma_s}^2 \mathrm{d}s} \frac{e^{\int_0^t \theta_s \mathrm{d}N_s}}{E_P\left(e^{\int_0^t \theta_s \mathrm{d}N_s} | \mathcal{G}_t\right)}.$$

As in Elliott and Osakwe (2006), $J_{it} = \int_0^t \langle J_s, e_i \rangle$ is the occupation time of state e_i up to time t and $\psi_j(u)$ is the Laplace exponent of $\hat{N}_1^j$:

$$\psi_j(u) = \ln E_P\left(e^{u\hat{N}_1^j}\right) = \hat{\lambda}_j \left(\frac{p_j \hat{\eta}_{1j}}{\hat{\eta}_{1j} - u} + \frac{q_j \hat{\eta}_{2j}}{\hat{\eta}_{2j} + u} - 1 \right).$$

Then, we have:

$$E_P\left(e^{\int_0^t \theta_s \mathrm{d}N_s} | \mathcal{G}_t\right) = e^{\sum_{i=1}^n J_{it} \psi_i(\hat{\theta}_i)},$$

which yields

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}|_{\mathcal{F}_t} \quad = e^{\int_0^t \theta_s \sigma_s \mathrm{d}W_s - \frac{1}{2}\int_0^t {\theta_s}^2 {\sigma_s}^2 \mathrm{d}s + \int_0^t \theta_s \mathrm{d}N_s - \sum_{i=1}^n J_{it} \psi_i(\hat{\theta}_i)}.$$

Denote $S_t = \frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}|_{\mathcal{F}_t}$. We have:

Lemma 1 S is an $(\mathcal{F}, P)$-martingale and the equivalent measure $\tilde{P}$ is well-defined.

Proof. For any $0 \leq u \leq v$,

$$E_P\left(\frac{S_v}{S_u} | \mathcal{F}_u\right) \quad = E_P\left(e^{\int_u^v \theta_s \sigma_s \mathrm{d}W_s - \frac{1}{2}\int_u^v {\theta_s}^2 {\sigma_s}^2 \mathrm{d}s} | \mathcal{F}_u\right) E_P\left(e^{\int_u^v \theta_s \mathrm{d}N_s - \sum_{i=1}^n (J_{iv} - J_{iu}) \psi_i(\hat{\theta}_i)} | \mathcal{F}_u\right)$$

$$= 1.$$

□

The vector $(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_n)$ should be chosen to make the discounted asset price process a martingale. Because the risk-free rate changes with the macroeconomic environment, we let the risk-free rate r satisfy $r_t = \langle \hat{r}, J_t \rangle$ where $\hat{r} = (\hat{r}_1, \hat{r}_2, ..., \hat{r}_n)$ denotes the vector of risk-free rates in all the regimes. Before presenting the martingale condition, we introduce the Laplace transform of the occupation times from Elliott and Osakwe (2006).

Proposition 1 For the $n-$state Markov switching model, the Laplace transform of the occupation times $Z = \{J_{1t}, J_{2t}, ..., J_{nt}\}$ is given by

$$\varphi(d) = E_P\left(e^{\langle d, Z\rangle}\right) = J_0{}' e^{\left(Q+\mathrm{diag}(d)\right)t}\mathbf{1}, \tag{1}$$

where $d = (d_1, d_2, ..., d_n)$, $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones, J_0 is initial state of the Markov chain J and Q is the corresponding generator matrix.

Then, we have:

Proposition 2 The martingale condition is satisfied if and only if

$$\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i) = 0 \quad \forall i = 1, 2, ..., N. \tag{2}$$

Proof.

$$\begin{aligned}
V_0 &= E_{\tilde{P}}\left(e^{-\int_0^t r_u \mathrm{d}u} V_t\right) \\
&= E_P\left(V_0 e^{\int_0^t (\mu_s - r_s)ds + \int_0^t (1+\theta_s)\sigma_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s{}^2\sigma_s{}^2 \mathrm{d}s + \int_0^t (1+\theta_s)\mathrm{d}N_s} e^{-\sum_{i=1}^n J_{it}\psi_i(\hat{\theta}_i)}\right) \\
&= E_P\left(V_0 e^{\sum_{i=1}^n J_{it}\left(\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)}\right) \\
&= V_0 J_0' e^{\left(Q+\mathrm{diag}\left(\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)\right)t}\mathbf{1}.
\end{aligned}$$

Let d be a null vector. Inserting it into Eq. (1), we readily have:

$$J_0' e^{Qt}\mathbf{1} = 1. \tag{3}$$

The sufficient condition of the martingale condition is then obvious. Denote

$$f(t) = J_0' e^{\left(Q+\mathrm{diag}\left(\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)\right)t}\mathbf{1}.$$

Assume that the martingale condition is satisfied:

$$f(t) = 1 \quad \forall t.$$

Then,

$$f'(0) = J_0'\left(Q + \mathrm{diag}\left(\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)\right)\mathbf{1} = 0.$$

Similarly, Eq. (3) implies

$$J_0'Q\mathbf{1} = 0.$$

Therefore,

$$J_0'\left(\mathrm{diag}\left(\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)\right)\mathbf{1} = 0.$$

Since this equation is satisfied for any initial vector of states J_0,

$$\hat{\mu}_i - \hat{r}_i + \frac{1}{2}\hat{\sigma}_i^2 + \hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(1+\hat{\theta}_i) - \psi_i(\hat{\theta}_i) = 0 \quad \forall i = 1, 2, ..., N.$$

□

The positive solutions $\{\hat{\theta}_i, i = 1, 2, ..., n\}$ can be solved from the n equations.

Proposition 3 The process $\{X_t; t \geq 0\}$ keeps a regime switching double exponential jump diffusion structure under $\tilde{P}$. Let X be defined under $\tilde{P}$ as follows:

$$X_t = \int_0^t \mu_s^* \mathrm{d}s + \int_0^t \sigma_s^* \mathrm{d}W_s^* + \int_0^t \mathrm{d}N_s^*.$$

Then, W^* defined by $W_t^* = W_t - \int_0^t \theta_s\sigma_s \mathrm{d}s$ is a standard Brownian motion, $\hat{\mu}_i^* = \hat{r}_i - \frac{1}{2}\hat{\sigma}_i^{*2} - \hat{\lambda}_i^*\left(\frac{p_i^*\hat{\eta}_{1i}^*}{\hat{\eta}_{1i}^* - 1} + \frac{(1-p_i^*)\hat{\eta}_{2i}^*}{\hat{\eta}_{2i}^* + 1} - 1\right)$, $\hat{\sigma}_i^* = \hat{\sigma}_i$, $\hat{\lambda}_i^* = \hat{\lambda}_i\omega_i$, $p_i^* = \frac{1}{\omega_i}\left(\frac{p_i\hat{\eta}_{1i}}{\hat{\eta}_{1i} - \hat{\theta}_i}\right)$, $\hat{\eta}_{1i}^* = \hat{\eta}_{1i} - \hat{\theta}_i$ and $\hat{\eta}_{2i}^* = \hat{\eta}_{2i} + \hat{\theta}_i$, where $\omega_i = E_P(e^{\hat{\theta}_i Y_i}) = \frac{p_i\hat{\eta}_{1i}}{\hat{\eta}_{1i} - \hat{\theta}_i} + \frac{(1-p_i)\hat{\eta}_{2i}}{\hat{\eta}_{2i} + \hat{\theta}_i}$.

Proof. Because

$$\begin{aligned}
E_{\tilde{P}}(e^{uX_t}) = \quad & E_P\left(V_0 e^{u\int_0^t \mu_s ds + \int_0^t (u+\theta_s)\sigma_s \mathrm{d}W_s - \frac{1}{2}\int_0^t {\theta_s}^2{\sigma_s}^2 \mathrm{d}s + \int_0^t (u+\theta_s)\mathrm{d}N_s} e^{-\sum_{i=1}^n J_{it}\psi_i(\hat{\theta}_i)}\right) \\
&= E_P\left(V_0 e^{\sum_{i=1}^n J_{it}\left(u\hat{\mu}_i + \frac{1}{2}u^2\hat{\sigma}_i^2 + u\hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(u+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)}\right) \\
&= {J_0}' e^{\left(Q + \mathrm{diag}\left(u\hat{\mu}_i + \frac{1}{2}u^2\hat{\sigma}_i^2 + u\hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(u+\hat{\theta}_i) - \psi_i(\hat{\theta}_i)\right)\right)t}\mathbf{1},
\end{aligned}$$

the Laplace exponent $\psi_i^*(u)$ of X^i under $\tilde{P}$ is

$$\psi_i^*(u) = u\hat{\mu}_i + \frac{1}{2}u^2\hat{\sigma}_i^2 + u\hat{\theta}_i\hat{\sigma}_i^2 + \psi_i(u+\hat{\theta}_i) - \psi_i(\hat{\theta}_i).$$

The term $\frac{1}{2}u^2\hat{\sigma}_i^2 + u\hat{\theta}_i\hat{\sigma}_i^2$ corresponds to $\hat{\sigma}_i W_1 \sim N(\hat{\theta}_i\hat{\sigma}_i^2, \hat{\sigma}_i^2)$. Then, we have that W^* defined by $W_t^* = W_t - \int_0^t \theta_s\sigma_s ds$ is a standard Brownian motion under $\tilde{P}$ with respect to $\mathcal{F}$.

Denote

$$\omega_i = \frac{p_i\hat{\eta}_{1i}}{\hat{\eta}_{1i} - \hat{\theta}_i} + \frac{(1-p_i)\hat{\eta}_{2i}}{\hat{\eta}_{2i} + \hat{\theta}_i}.$$

After simple computations, we obtain:

$$\psi_i^*(u) = \left(\hat{\mu}_i + \hat{\theta}_i\hat{\sigma}_i^2\right)u + \frac{1}{2}\hat{\sigma}_i^2u^2 + \hat{\lambda}_i\omega_i\left(\frac{1}{\omega_i}\left(\frac{p_i\hat{\eta}_{1i}}{\hat{\eta}_{1i} - \hat{\theta}_i}\right)\frac{\hat{\eta}_{1i} - \hat{\theta}_i}{\hat{\eta}_{1i} - \hat{\theta}_i - u} + \frac{1}{\omega_i}\left(\frac{(1-p_i)\hat{\eta}_{2i}}{\hat{\eta}_{2i} + \hat{\theta}_i}\right)\frac{\hat{\eta}_{2i} + \hat{\theta}_i}{\hat{\eta}_{2i} + \hat{\theta}_i + u}\right).$$

This expression shows that X^i remains a double exponential jump diffusion process under $\tilde{P}$ where

$$\begin{cases} \hat{\mu}_i^* = \hat{\mu}_i + \hat{\theta}_i\hat{\sigma}_i^2 \\ \hat{\sigma}_i^* = \hat{\sigma}_i \\ \hat{\lambda}_i^* = \hat{\lambda}_i\omega_i \\ p_i^* = \frac{1}{\omega_i}\left(\frac{p_i\hat{\eta}_{1i}}{\hat{\eta}_{1i} - \hat{\theta}_i}\right) \\ \hat{\eta}_{1i}^* = \hat{\eta}_{1i} - \hat{\theta}_i \\ \hat{\eta}_{2i}^* = \hat{\eta}_{2i} + \hat{\theta}_i. \end{cases}$$

From $E_{\tilde{P}}(e^{X_t}) = e^{rt}$, we see that $\hat{\mu}_i^*$ satisfies

$$\hat{\mu}_i^* = \hat{r}_i - \frac{1}{2}\hat{\sigma}_i^{*2} - \hat{\lambda}_i^*\Big(\frac{p_i^*\hat{\eta}_{1i}^*}{\hat{\eta}_{1i}^* - 1} + \frac{(1-p_i^*)\hat{\eta}_{2i}^*}{\hat{\eta}_{2i}^* + 1} - 1\Big).$$

Therefore, the structure of the regime switching double exponential jump diffusion process is unchanged under the new measure $\tilde{P}$. □

3 The First Passage Time of The Regime Switching Jump Diffusion Process

The first passage time problem across a constant level is related to the up-crossing and down-crossing ladder processes of the fluid embedding of X. Denote as $A = \{A_t; t \geq 0\}$ the fluid embedding of X. Contrary to X, A is a continuous process. Its paths are constructed from the paths of X by replacing positive jumps by linear segments with slope +1 and negative jumps by linear segments with slope -1. We define an irreducible continuous time Markov chain process $Y = \{Y_t; t \geq 0\}$ with a finite state space $E = E^+ \cup E^0 \cup E^-$. The spaces E^0, E^+ and E^- correspond to the states where X moves as a pure diffusion, makes a positive jump and makes a negative jump, respectively. Conditional on this enlarged Markov chain, A is a Brownian motion with a drift that includes linear segments of slope +1 and -1.

Specifically, A is represented as follows:

$$A_t = A_0 + \int_0^t u(Y_s)\mathrm{d}s + \int_0^t v(Y_s)\mathrm{d}W_s$$

where

$$u(j) = \begin{cases} 1 & \text{if } j \in E^+ \\ \hat{\mu}_j & \text{if } j \in E^0 \\ -1 & \text{if } j \in E^- \end{cases} \qquad \text{and} \qquad v(j) = \begin{cases} \hat{\sigma}_j & \text{if } j \in E^0 \\ 0 & \text{otherwise.} \end{cases}$$

The up-crossing and down-crossing ladder processes $\tilde{Y}^+$, $\tilde{Y}^-$ of (A, Y) are defined as follows:

$$\tilde{Y}_z^+ = Y(\tau_z^+) \qquad \text{and} \qquad \tilde{Y}_z^- = Y(\tau_z^-)$$

where

$$\tau_z^+ = \inf\{s \geq 0 : A_s > z\} \qquad \text{and} \qquad \tau_z^- = \inf\{s \geq 0 : A_s < z\}.$$

They are Markov processes with state spaces $E^0 \cup E^+$ and $E^0 \cup E^-$, respectively. Denote as Q_+ and Q_- the generator matrices of $\tilde{Y}^+$ and $\tilde{Y}^-$. The corresponding initial distributions are $n \times 2n$ matrices ζ^+ and ζ^- defined as follows:

$$\zeta^+(i,j) = P_{0,i}(\tilde{Y}_0^+ = j) \qquad \forall i \in E^-, j \in E^+ \cup E^0 \tag{4}$$

and

$$\zeta^-(i,j) = P_{0,i}(\tilde{Y}_0^- = j) \qquad \forall i \in E^+, j \in E^- \cup E^0. \tag{5}$$

where $P_{0,i}(\cdot) = P(\cdot | X_0 = 0, J_0 = e_i)$. The generator matrices Q_+ and Q_- of $\tilde{Y}^+$ and $\tilde{Y}^-$ are related to the Wiener-Hopf factorization of (A, Y). In our case, the Wiener-Hopf factorization of (A, Y) is as follows:

Definition 1 The pair of irreducible $2n * 2n$ matrices $(Q^{(\hat{a},-)}, Q^{(\hat{a},+)})$ with non-negative off-diagonal elements and non-positive row sums is called the Wiener-Hopf factorization of (A, Y) if

$$\Xi(-Q^{(\hat{a},+)}, W^+) = \Xi(Q^{(\hat{a},-)}, W^-) = O,$$

where

$$\Xi(S, W) = \frac{1}{2}\Sigma^2 W S^2 + VWS + Q_{\hat{a}} W,$$

with the $3n \times 3n$ diagonal matrices:

$$\Sigma = \begin{pmatrix} O_n & & \\ & \begin{pmatrix} \hat{\sigma}_1 & & \\ & \ddots & \\ & & \hat{\sigma}_n \end{pmatrix} & \\ & & O_n \end{pmatrix}, \qquad V = \begin{pmatrix} I_n & & \\ & \begin{pmatrix} \hat{\mu}_1 & & \\ & \ddots & \\ & & \hat{\mu}_n \end{pmatrix} & \\ & & -I_n \end{pmatrix},$$

and the $3n \times 3n$ matrix:

$$Q_{\hat{a}} = \begin{pmatrix} T^1 & t^1 & O_n \\ B^+ & Q - D_{\hat{a}} & B^- \\ O_n & t^2 & T^2 \end{pmatrix}.$$

In this representation, the $n \times n$ matrix Q is the generator of J, $\hat{a} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_n) > 0$, $D_{\hat{a}} = \text{diag}(\lambda_i + \hat{a}_i)$, O_n is a zero matrix of size $n \times n$ and I_n is an identity matrix of size $n \times n$. We also have:

$$B^+ = \begin{pmatrix} \hat{\lambda}_1 p_1 & & \\ & \ddots & \\ & & \hat{\lambda}_n p_n \end{pmatrix}, \qquad B^- = \begin{pmatrix} \hat{\lambda}_1(1-p_1) & & \\ & \ddots & \\ & & \hat{\lambda}_n(1-p_n) \end{pmatrix}$$

and

$$T^i = \begin{pmatrix} -\hat{\eta}_{i1} & & \\ & \ddots & \\ & & -\hat{\eta}_{in} \end{pmatrix}, \qquad t^i = \begin{pmatrix} \hat{\eta}_{i1} & & \\ & \ddots & \\ & & \hat{\eta}_{in} \end{pmatrix}.$$

Finally, we need the $3n \times 2n$ matrices:

$$W^+ = \begin{pmatrix} I_{2n} \\ \zeta^+ \end{pmatrix} \qquad W^- = \begin{pmatrix} \zeta^- \\ I_{2n} \end{pmatrix}$$

where I_{2n} is an identity matrix of size $2n \times 2n$ and ζ^+ and ζ^- are defined in Eqs (4) and (5).

Jiang and Pistorius (2008) have established the direct relationship between the first passage time of X and the matrix Wiener-Hopf factorization of (A, Y) and the following proposition is a consequence of their Theorem 3.

Proposition 4 Denote as τ the first passage time of X below a constant level b as

$$\tau = \inf\{t > 0 : X_t \leq b\}$$

and the contingent payoff $h(\tau) = \langle J_\tau, \hat{h} \rangle$ where $\hat{h} = (\hat{h}_1, ..., \hat{h}_n)$. Denote $a_t = \langle \hat{a}, J_t \rangle$, then

$$E\left(e^{-\int_0^\tau a_s \mathrm{d}s + wX_\tau} h(\tau) \right) = Y_0' W^- e^{Q^{(\hat{a},-)}(x-b)+wb} \tilde{h}$$

where w is a constant, x is the initial point of X, Y_0 is the initial state of Y,

$$\tilde{h} = \left(\left(\hat{h}_1, ..., \hat{h}_n \right), \left(\frac{\eta_{21}}{w + \eta_{21}} \hat{h}_1, ..., \frac{\eta_{2n}}{w + \eta_{2n}} \hat{h}_n \right) \right)',$$

and $Q^{(\hat{a},-)}$ is the Wiener-Hopf factor defined above.

We now present a numerical method allowing us to compute W^- and $Q^{(\hat{a},-)}$. Let ϑ be an eigenvector of $Q^{(\hat{a},-)}$ and let β be its associated eigenvalue. Right-multiply by ϑ the following equation:

$$\Xi(Q^{(\hat{a},-)}, W^-) = \frac{1}{2}\Sigma^2 W^- {Q^{(\hat{a},-)}}^2 + V W^- Q^{(\hat{a},-)} + Q_{\hat{a}} W^- = 0,$$

and obtain:

$$\frac{1}{2}\Sigma^2 W^- \beta^2 \vartheta + V W^- \beta \vartheta + Q_{\hat{a}} W^- \vartheta = 0,$$

or equivalently:

$$\left(\frac{1}{2}\Sigma^2 \beta^2 + V\beta + Q_{\hat{a}}\right) W^- \vartheta = 0.$$

Denote $K(\beta) = \frac{1}{2}\Sigma^2\beta^2 + V\beta + Q_{\hat{a}}$. The matrix $K(\beta)$ is singular and $f(\beta) = \det(K(\beta)) = 0$. From Barlow, Rogers, and Williams (1980) and Rogers and Shi (1994), we have the following lemma:

Lemma 2 Suppose that $\hat{a} > 0$. The equation $f(\beta) = 0$ has $4n$ different roots $\{\beta_i, i = 1, 2, ..., 4n\}$ and the roots are ranked as follows:

$$Re(\beta_1) \leq Re(\beta_2) \leq ... \leq Re(\beta_{2n}) < 0 < Re(\beta_{2n+1}) \leq Re(\beta_{2n+2}) \leq ... \leq Re(\beta_{4n}).$$

Then, $Q^{(\hat{a},-)}$ has $2n$ distinct eigenvalues $\{\beta_i, i = 1, 2, ..., 2n\}$.

Denote as ϑ_i the eigenvector of $Q^{(\hat{a},-)}$ corresponding to the eigenvalue β_i and

$$\gamma_i = W^- \vartheta_i = (\gamma_{i,1}, ..., \gamma_{i,3n})'.$$

We can compute γ_i by solving a system of linear equations $K(\beta_i)\gamma_i = 0$. Then, the structure of W^- gives

$$\vartheta_i = (\gamma_{i,n+1}, ..., \gamma_{i,3n})' \qquad i = 1, 2, .., 2n$$

and

$$\zeta^- \vartheta_i = (\gamma_{i,1}, ..., \gamma_{i,n})' \qquad i = 1, 2, ..., 2n.$$

Denote as ζ_k^- the k^{th} row of ζ^-. The latter linear equations give ζ^- by rearranging the problem into a system of linear equations as follows:

$$Z' \zeta_k^- = (\gamma_{1,k}, ..., \gamma_{2n,k})' \qquad k = 1, 2, ..., n,$$

where $Z = [\vartheta_1, \vartheta_2, ..., \vartheta_{2n}]$.

Using standard algebra, we have:

$$Q^{(\hat{a},-)} = Z \text{ diag}\{\beta_1, \beta_2, ..., \beta_{2n}\} \ Z^{-1}.$$

The matrix exponential is then computed as

$$e^{Q^{(\hat{a},-)}x} = Z \text{ diag}\{e^{\beta_1 x}, e^{\beta_2 x}, ..., e^{\beta_{2n} x}\} \ Z^{-1}.$$

4 The Payoff Structure of Different Instruments in Capital Structure

The capital structure of the bank is assumed to include equity, CoCo bonds, straight bonds, deposits and deposit insurance. The straight bonds have a face value of D_1, pay a continuous coupon rate of c_1, and have a maturity profile of $\psi_1(t) = me^{-mt}$. The face value and the continuous coupon rate of deposits are D_2 and c_2, respectively. The face value of CoCo bonds is D_3 and continuous coupon rate paid before conversion is c_3. The maturity profiles of deposit and CoCo bonds are $\psi_2(t) = ke^{-kt}$ and $\psi_3(t) = le^{-lt}$, respectively. The bank pays the deposit insurance premium DI to guarantee that the deposit will be fully paid if it is liquidated. The conversion of CoCos is triggered when the asset value falls below the proportion α of the full debt value. The conversion time τ_1 is defined as follows:

$$\tau_1 = \inf\{t \geq 0 : V_t < \alpha(D_1 + D_2 + D_3)\}.$$

We assume that default can only occur after conversion. The default time τ_2 is defined as follows

$$\tau_2 = \inf\{t \geq 0 : V_t < \alpha(D_1 + D_2)\}.$$

The bank is assumed to generate between any two times t and $t + \mathrm{d}t$ a stream $\delta V_t \mathrm{d}t$ of cash flows proportional to the bank asset value, with $\delta \in (0, 1)$. These cash flows are used to service the bank debt and to pay dividends to shareholders. The tax rate is γ and the dividend $(1-\gamma)(\delta V_t - c_1 D_1 - c_2 D_2 - c_3 D_3)$ is paid to shareholders before the conversion of CoCo bonds. However, after conversion, the dividend $(1 - \gamma)(\delta V_t - c_1 D_1 - c_2 D_2)$ is paid to the original shareholders and the new shareholders. The proportion of original shareholders is assumed to be ρ after conversion. The dividend allocated to the original shareholders and the new shareholders is $\rho(1-\gamma)(\delta V_t - c_1 D_1 - c_2 D_2)$ and $(1-\rho)(1-\gamma)(\delta V_t - c_1 D_1 - c_2 D_2)$, respectively. After conversion, the pricing of these financial instruments is conducted under the risk-neutral measure $\tilde{P}$. The price of these financial instruments is obtained using propositions 1 and 4 and the decomposition:

$$E\left(\int_0^{\tau} e^{-\int_0^s r_u \mathrm{d}u} \mathrm{d}s\right) = E\left(\int_0^{\infty} e^{-\int_0^s r_u \mathrm{d}u} \mathrm{d}s\right) - E\left(e^{-\int_0^{\tau} r_u \mathrm{d}u} E\Big(\int_{\tau}^{\infty} e^{-\int_{\tau}^s r_u \mathrm{d}u} \mathrm{d}s | \mathcal{F}_{\tau}\Big)\right).$$

We also compute integrals of matrix exponentials as follows:

$$\int_0^T e^{At} \mathrm{d}t = A^{-1}(e^{AT} - I),$$

where A is a nonsingular matrix and I is the identity matrix.

For the sake of subsequent computations, we define the following $2n \times n$ matrix $\tilde{H}$, $2n \times 3n$ matrix $\tilde{K}$ and $2n$ vector $\tilde{I}$:

$$\tilde{H}(w) = \begin{pmatrix} I_n \\ \begin{pmatrix} \frac{\hat{\eta}_{21}}{w+\hat{\eta}_{21}} & & \\ & \ddots & \\ & & \frac{\hat{\eta}_{2n}}{w+\hat{\eta}_{2n}} \end{pmatrix} \end{pmatrix}$$

and

$$\tilde{K}(w) = \begin{pmatrix} O_n & I_n & (O) \\ (O) & & \begin{pmatrix} \frac{\hat{\eta}_{21}}{w+\hat{\eta}_{21}} & & \\ & \ddots & \\ & & \frac{\hat{\eta}_{2n}}{w+\hat{\eta}_{2n}} \end{pmatrix} \end{pmatrix}$$

and

$$\tilde{I} = \begin{pmatrix} I_n \\ \begin{pmatrix} \frac{\hat{\eta}_{21}}{1+\hat{\eta}_{21}} \\ \vdots \\ \frac{\hat{\eta}_{2n}}{1+\hat{\eta}_{2n}} \end{pmatrix} \end{pmatrix}.$$

4.1 Bank Equity and Deposit Insurance

Before conversion, the market value of the shareholders' equity is denoted by S and satisfies:

$$S = E\left(\int_0^{\tau_1} e^{-\int_0^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2 - c_3 D_3\Big)\mathrm{d}s + \rho e^{-\int_0^{\tau_1} r_u \mathrm{d}u} S_{\tau_1}\right),$$

where

$$S_{\tau_1} = E\left(\int_{\tau_1}^{\tau_2} e^{-\int_{\tau_1}^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2\Big)\mathrm{d}s + e^{-\int_{\tau_1}^{\tau_2} r_u \mathrm{d}u} \pi_1 V_{\tau_2} | \mathcal{F}_{\tau_1}\right),$$

and where π_1 is the constant recovery rate of asset value for the distribution to shareholders after liquidation.

Then,

$$
\begin{aligned}
S &= E\left(\int_0^\infty e^{-\int_0^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2 - c_3 D_3\Big)\mathrm{d}s\right)\\
&\quad -E\left(e^{-\int_0^{\tau_1} r_u \mathrm{d}u} E\Big(\int_{\tau_1}^\infty e^{-\int_{\tau_1}^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2 - c_3 D_3\Big)\mathrm{d}s|\mathcal{F}_{\tau_1}\Big)\right)\\
&\quad +E\left(\rho e^{-\int_0^{\tau_1} r_u \mathrm{d}u} E\Big(\int_{\tau_1}^\infty e^{-\int_{\tau_1}^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2\Big)\mathrm{d}s + e^{-\int_{\tau_1}^{\tau_2} r_u \mathrm{d}u}\pi_1 V_{\tau_2}|\mathcal{F}_{\tau_1}\Big)\right)\\
&\quad -E\left(\rho e^{-\int_0^{\tau_2} r_u \mathrm{d}u} E\Big(\int_{\tau_2}^\infty e^{-\int_{\tau_2}^s r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1 D_1 - c_2 D_2\Big)\mathrm{d}s|\mathcal{F}_{\tau_2}\Big)\right)\\
&= S_1 - S_2 + S_3 - S_4,
\end{aligned}
$$

where S_1, S_2, S_3, S_4 represent the first, second, third and fourth term, respectively. We first compute:

$$
\begin{aligned}
S_1 &= \int_0^\infty \delta\Big(1-\gamma\Big)V_0 {J_0}' e^{\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)s}\mathbf{1}\mathrm{d}s\\
&\quad -\int_0^\infty \Big(1-\gamma\Big)\Big(c_1 D_1 + c_2 D_2 + c_3 D_3\Big){J_0}' e^{\Big(Q-\mathrm{diag}\big(\hat{r}\big)\Big)s}\mathbf{1}\mathrm{d}s,
\end{aligned}
$$

so that

$$
\begin{aligned}
S_1 &= \Big(1-\gamma\Big)\Big(c_1 D_1 + c_2 D_2 + c_3 D_3\Big){J_0}'\Big(Q - \mathrm{diag}(\hat{r})\Big)^{-1}\mathbf{1}\\
&\quad -\Big(1-\gamma\Big)\delta V_0 {J_0}'\Big(Q + \mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}.
\end{aligned}
$$

Then, we have:

$$
\begin{aligned}
S_2 &= E\left(e^{-\int_0^{\tau_1} r_u \mathrm{d}u}\int_{\tau_1}^\infty \Big(1-\gamma\Big)\delta V_{\tau_1} J'_{\tau_1} e^{\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)\Big(s-\tau_1\Big)}\mathbf{1}\mathrm{d}s\right)\\
&\quad -E\left(e^{-\int_0^{\tau_1} r_u \mathrm{d}u}\int_{\tau_1}^\infty \Big(1-\gamma\Big)\Big(c_1 D_1 + c_2 D_2 + c_3 D_3\Big)J'_{\tau_1} e^{\Big(Q-\mathrm{diag}\big(\hat{r}\big)\Big)\Big(s-\tau_1\Big)}\mathbf{1}\mathrm{d}s\right)
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 &= E\left(e^{-\int_0^{\tau_1} r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(c_1 D_1 + c_2 D_2 + c_3 D_3\Big)J'_{\tau_1}\Big(Q - \mathrm{diag}(\hat{r})\Big)^{-1}\mathbf{1}\right)\\
&\quad -E\left(e^{-\int_0^{\tau_1} r_u \mathrm{d}u}\Big(1-\gamma\Big)\delta V_{\tau_1} J'_{\tau_1}\Big(Q + \mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}\right)
\end{aligned}
$$

and also

$$
\begin{aligned}
S_2 &= \Big(1-\gamma\Big)\Big(c_1 D_1 + c_2 D_2 + c_3 D_3\Big)Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\dfrac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde{H}(0)\Big(Q - \mathrm{diag}\big(\hat{r}\big)\Big)^{-1}\mathbf{1}\\
&\quad -\Big(1-\gamma\Big)\delta V_0 Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\dfrac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde{H}(1)\Big(Q + \mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}.
\end{aligned}
$$

Further, we obtain:

$$\begin{aligned}
S_3 &= E\left(\rho e^{-\int\limits_0^{\tau_1} r_u \mathrm{d}u} \int\limits_{\tau_1}^{\infty} \Big(1-\gamma\Big)\delta V_{\tau_1} J'_{\tau_1} e^{\left(Q+\mathrm{diag}(\varphi(1)-\hat{r})\right)\left(s-\tau_1\right)} \mathbf{1}\mathrm{d}s\right)\\
&-E\left(\rho e^{-\int\limits_0^{\tau_1} r_u \mathrm{d}u} \int\limits_{\tau_1}^{\infty} \Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big) J'_{\tau_1} e^{\left(Q-\mathrm{diag}(\hat{r})\right)\left(s-\tau_1\right)} \mathbf{1}\mathrm{d}s\right) + E\left(\rho e^{-\int\limits_0^{\tau_2} r_u \mathrm{d}u} \pi_1 V_{\tau_2}\right)
\end{aligned}$$

and

$$\begin{aligned}
S_3 &= E\left(\rho e^{-\int\limits_0^{\tau_1} r_u \mathrm{d}u} \Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big) J'_{\tau_1}\Big(Q-\mathrm{diag}(\hat{r})\Big)^{-1}\mathbf{1}\right)\\
&-E\left(\rho e^{-\int\limits_0^{\tau_1} r_u \mathrm{d}u} \Big(1-\gamma\Big)\delta V_{\tau_1} J'_{\tau_1}\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}\right)\\
&+\rho\pi_1 V_0 Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{I}
\end{aligned}$$

and then

$$\begin{aligned}
S_3 &= \rho\Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big)Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde{H}(0)\Big(Q-\mathrm{diag}(\hat{r})\Big)^{-1}\mathbf{1}\\
&-\rho\Big(1-\gamma\Big)\delta V_0 Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde{H}(1)\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}\\
&+\rho\pi_1 V_0 Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{I}.
\end{aligned}$$

Finally,

$$S_4 = E\left(\rho e^{-\int\limits_0^{\tau_2} r_u \mathrm{d}u} E\Big(\int\limits_{\tau_2}^{\infty} e^{-\int\limits_{\tau_2}^{s} r_u \mathrm{d}u}\Big(1-\gamma\Big)\Big(\delta V_s - c_1D_1 - c_2D_2\Big)\mathrm{d}s|\mathcal{F}_{\tau_2}\Big)\right)$$

gives

$$\begin{aligned}
S_4 &= E\left(\rho e^{-\int\limits_0^{\tau_2} r_u \mathrm{d}u} \Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big) J'_{\tau_2}\Big(Q-\mathrm{diag}(\hat{r})\Big)^{-1}\mathbf{1}\right)\\
&-E\left(\rho e^{-\int\limits_0^{\tau_2} r_u \mathrm{d}u} \Big(1-\gamma\Big)\delta V_{\tau_2} J'_{\tau_2}\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}\right),
\end{aligned}$$

so that

$$\begin{aligned}
S_4 &= \rho\Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big)Y'_0 W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{H}(0)\big(Q-\mathrm{diag}(\hat{r})\big)^{-1}\mathbf{1}\\
&-\rho\Big(1-\gamma\Big)\delta V_0 {Y_0}' W^- e^{Q^{(\hat{r},-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{H}(1)\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat{r}\big)\Big)^{-1}\mathbf{1}.
\end{aligned}$$

4.2 Straight Bonds and Deposits

The market values of straight bonds and deposits with face value 1 maturing at t are expressed as follows

$$B(t) = E\left(\int_0^{\min(t,\tau_2)} e^{-\int_0^s r_u \mathrm{d}u} c_1 \mathrm{d}s + \mathbb{1}_{\{t\geq\tau_2\}} e^{-\int_0^{\tau_2} r_u \mathrm{d}u} \frac{\pi_2 V_{\tau_2}}{D_1+D_2} + \mathbb{1}_{\{t<\tau_2\}} e^{-\int_0^t r_u \mathrm{d}u}\right)$$

and

$$D(t) = E\left(\int_0^{\min(t,\tau_2)} e^{-\int_0^s r_u \mathrm{d}u} c_2 \mathrm{d}s + \mathbb{1}_{\{t\geq\tau_2\}} e^{-\int_0^{\tau_2} r_u \mathrm{d}u} \frac{\pi_2 V_{\tau_2}}{D_1+D_2} + \mathbb{1}_{\{t<\tau_2\}} e^{-\int_0^t r_u \mathrm{d}u}\right),$$

where π_2 is the constant recovery rate of asset value for the distribution to debt holders after liquidation. Then, the total value of straight bonds is

$$\begin{aligned}
B &= D_1 \int_0^\infty m e^{-mt} B(t)\mathrm{d}t \\
&= D_1 \int_0^\infty m e^{-mt} E\left(\int_0^{\min(t,\tau_2)} e^{-\int_0^s r_u \mathrm{d}u} c_1 \mathrm{d}s + \mathbb{1}_{\{t\geq\tau_2\}} e^{-\int_0^{\tau_2} r_u \mathrm{d}u} \frac{\pi_2 V_{\tau_2}}{D_1+D_2} + \mathbb{1}_{\{t<\tau_2\}} e^{-\int_0^t r_u \mathrm{d}u}\right)\mathrm{d}t \\
&= D_1 E\left(\int_0^\infty m e^{-mt} \int_0^{\min(t,\tau_2)} e^{-\int_0^s r_u \mathrm{d}u} c_1 \mathrm{d}s\mathrm{d}t\right) + D_1 E\left(\int_0^\infty m e^{-mt} \mathbb{1}_{\{t\geq\tau_2\}} e^{-\int_0^{\tau_2} r_u \mathrm{d}u} \frac{\pi_2 V_{\tau_2}}{D_1+D_2}\mathrm{d}t\right) \\
&\quad + D_1 E\left(\int_0^\infty m e^{-mt} \mathbb{1}_{\{t<\tau_2\}} e^{-\int_0^t r_u \mathrm{d}u}\mathrm{d}t\right) \\
&= D_1 E\left(\int_0^{\tau_2} m c_1 e^{-\int_0^s r_u \mathrm{d}u} \int_s^\infty e^{-mt}\mathrm{d}t\mathrm{d}s\right) + \frac{\pi_2 D_1}{D_1+D_2} E\left(e^{-\int_0^{\tau_2}(r_u+m)\mathrm{d}u} V_{\tau_2}\right) \\
&\quad + m D_1 E\left(\int_0^{\tau_2} e^{-\int_0^t (r_u+m)\mathrm{d}u}\mathrm{d}t\right) \\
&= \underbrace{(c_1+m) D_1 E\left(\int_0^{\tau_2} e^{-\int_0^s (r_u+m)\mathrm{d}u}\mathrm{d}s\right)}_{B_1} + \underbrace{\frac{\pi_2 D_1}{D_1+D_2} E\left(e^{-\int_0^{\tau_2}(r_u+m)\mathrm{d}u} V_{\tau_2}\right)}_{B_2}.
\end{aligned}$$

We exchange the order of integration for t and s and obtain

$$D_1 E\left(\int_0^\infty m e^{-mt} \int_0^{\min(t,\tau_2)} e^{-\int_0^s r_u \mathrm{d}u} c_1 \mathrm{d}s\mathrm{d}t\right) = D_1 E\left(\int_0^{\tau_2} m c_1 e^{-\int_0^s r_u \mathrm{d}u} \int_s^\infty e^{-mt}\mathrm{d}t\mathrm{d}s\right).$$

We can now compute:

$$
\begin{aligned}
B_1 &= (c_1+m)D_1 E\left(\int_0^{\tau_2} e^{-\int_0^s (r_u+m)\mathrm{d}u}\mathrm{d}s\right)\\
&= (c_1+m)D_1\Bigg(E\Big(\int_0^{\infty} e^{-\int_0^s (r_u+m)\mathrm{d}u}\mathrm{d}s\Big) - E\Big(e^{-\int_0^{\tau_2}(r_u+m)\mathrm{d}u} E\Big(\int_{\tau_2}^{\infty} e^{-\int_{\tau_2}^{s}(r_u+m)\mathrm{d}u}\mathrm{d}s|\mathcal{F}_{\tau_2}\Big)\Big)\Bigg)\\
&= (c_1+m)D_1\Bigg(\Big(-J_0'\Big(Q-\mathrm{diag}(\hat{r}+m)\Big)^{-1}\mathbf{1}\Big) + E\Big(e^{-\int_0^{\tau_2}(r_u+m)\mathrm{d}u} J_{\tau_2}'\Big(Q-\mathrm{diag}(\hat{r}+m)\Big)^{-1}\mathbf{1}\Big)\Bigg)\\
&= (c_1+m)D_1\Bigg(Y_0'W^- e^{Q^{(\hat{r}+m,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{H}(0)\Big(Q-\mathrm{diag}(\hat{r}+m)\Big)^{-1}\mathbf{1}\\
&\quad -J_0'\Big(Q-\mathrm{diag}(\hat{r}+m)\Big)^{-1}\mathbf{1}\Bigg),
\end{aligned}
$$

and

$$
B_2 = \frac{\pi_2 D_1}{D_1+D_2} E\left(e^{-\int_0^{\tau_2}\left(r_u+m\right)\mathrm{d}u} V_{\tau_2}\right) = \frac{\pi_2 D_1}{D_1+D_2} V_0 Y_0' W^- e^{Q^{\left(\hat{r}+m,-\right)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{I}.
$$

The total value of deposits is computed in the same way:

$$
\begin{aligned}
D &= D_2\int_0^{\infty} ke^{-kt}D(t)\mathrm{d}t\\
&= (c_2+k)D_2\Bigg(Y_0'W^- e^{Q^{(\hat{r}+k,-)}\left(x-\ln\frac{\alpha\,(D_1+D_2)}{V_0}\right)}\tilde{H}(0)\Big(Q-\mathrm{diag}(\hat{r}+k)\Big)^{-1}\mathbf{1}\\
&\quad -J_0'\Big(Q-\mathrm{diag}(\hat{r}+k)\Big)^{-1}\mathbf{1}\Bigg) + \frac{\pi_2 D_2}{D_1+D_2}V_0 Y_0' W^- e^{Q^{\left(\hat{r}+k,-\right)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{I}.
\end{aligned}
$$

4.3 Deposit Insurance

The market value of deposit insurance is

$$
DI = E\left(e^{-\int_0^{\tau_2} r_u\mathrm{d}u}\left[E\left(\int_0^{\infty} ke^{-kt}\left(\int_{\tau_2}^{\tau_2+t} e^{-\int_{\tau_2}^{s} r_u\mathrm{d}u} c_2D_2\mathrm{d}s + e^{-\int_{\tau_2}^{\tau_2+t} r_u\mathrm{d}u} D_2\right)\mathrm{d}t - \pi_2\frac{D_2}{D_1+D_2}V_{\tau_2}|\mathcal{F}_{\tau_2}\right)\right]^+\right).
$$

We first compute:

$$
\begin{aligned}
E\left(\int_0^{\infty} ke^{-kt}\int_{\tau_2}^{\tau_2+t} e^{-\int_{\tau_2}^{s} r_u\mathrm{d}u} c_2D_2\mathrm{d}s\mathrm{d}t|\mathcal{F}_{\tau_2}\right) &= E\left(kc_2D_2\int_{\tau_2}^{\infty}\int_{s-\tau_2}^{\infty} e^{-kt}e^{-\int_{\tau_2}^{s} r_u\mathrm{d}u}\mathrm{d}t\mathrm{d}s|\mathcal{F}_{\tau_2}\right)\\
&= c_2D_2E\left(\int_{\tau_2}^{\infty} e^{-\int_{\tau_2}^{s}\left(k+r_u\right)\mathrm{d}u}\mathrm{d}s|\mathcal{F}_{\tau_2}\right)\\
&= -c_2D_2J_{\tau_2}'\Big(Q-\mathrm{diag}(k+\hat{r})\Big)^{-1}\mathbf{1}
\end{aligned}
$$

and

$$E\left(\int_0^\infty ke^{-kt}e^{-\int_{\tau_2}^{\tau_2+t} r_u\mathrm{d}u}D_2\mathrm{d}t|\mathcal{F}_{\tau_2}\right) = \int_0^\infty kD_2J_{\tau_2}e^{\left(Q-\mathrm{diag}\left(k+\hat{r}\right)\right)t}\mathbf{1}\mathrm{d}t$$
$$= -kD_2J'_{\tau_2}\Big(Q-\mathrm{diag}(k+\hat{r})\Big)^{-1}\mathbf{1}.$$

Then,

$$DI \quad = E\left(e^{-\int_0^{\tau_2} r_u\mathrm{d}u}\Big(-(c_2+k)D_2J'_{\tau_2}\Big(Q-\mathrm{diag}(k+\hat{r})\Big)^{-1}\mathbf{1}-\pi_2\frac{D_2}{D_1+D_2}V_{\tau_2}\Big)^+\right)$$

and

$$DI \quad = \Bigg(-(c_2+k)D_2Y_0'W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{H}(0)\Big(Q-\mathrm{diag}(k+\hat{r})\Big)^{-1}\mathbf{1}$$
$$-\pi_2\frac{D_2}{D_1+D_2}V_0Y_0'W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde{I}\Bigg)^+.$$

4.4 CoCos

The market value of CoCo bonds with face value 1 maturing at t is

$$C(t) \quad = E\left(\int_0^{\min(t,\tau_1)} e^{-\int_0^s r_u\mathrm{d}u}c_3\mathrm{d}s + \frac{1-\rho}{D_3}\mathbb{1}_{\{\tau_1\le t\}}e^{-\int_0^{\tau_1} r_u\mathrm{d}u}S_{\tau_1} + \mathbb{1}_{\{\tau_1>t\}}e^{-\int_0^t r_u\mathrm{d}u}\right).$$

Then, the value of all CoCo bonds is

$$\begin{aligned}
C \quad &= D_3\int_0^\infty le^{-lt}C(t)\mathrm{d}t \\
&= D_3\int_0^\infty le^{-lt}E\left(\int_0^{\min(t,\tau_1)} e^{-\int_0^s r_u\mathrm{d}u}c_3\mathrm{d}s + \frac{1-\rho}{D_3}\mathbb{1}_{\{\tau_1\le t\}}e^{-\int_0^{\tau_1} r_u\mathrm{d}u}S_{\tau_1} + \mathbb{1}_{\{\tau_1>t\}}e^{-\int_0^t r_u\mathrm{d}u}\right)\mathrm{d}t \\
&= D_3E\left(\int_0^\infty le^{-lt}\int_0^{\min(t,\tau_1)} e^{-\int_0^s r_u\mathrm{d}u}c_3\mathrm{d}s\mathrm{d}t\right) + D_3E\left(\int_0^\infty le^{-lt}\frac{1-\rho}{D_3}\mathbb{1}_{\{\tau_1\le t\}}e^{-\int_0^{\tau_1} r_u\mathrm{d}u}S_{\tau_1}\mathrm{d}t\right) \\
&\quad +D_3E\left(\int_0^\infty le^{-lt}\mathbb{1}_{\{\tau_1>t\}}e^{-\int_0^t r_u\mathrm{d}u}\mathrm{d}t\right) \\
&= c_3D_3E\left(\int_0^{\tau_1} e^{-\int_0^s (l+r_u)\mathrm{d}u}\mathrm{d}s\right) + \Big(1-\rho\Big)E\left(e^{-\int_0^{\tau_1}(l+r_u)\mathrm{d}u}S_{\tau_1}\right) + lD_3E\left(\int_0^{\tau_1} e^{-\int_0^t (r_u+l)\mathrm{d}u}\mathrm{d}t\right) \\
&= \underbrace{D_3(c_3+l)E\left(\int_0^{\tau_1} e^{-\int_0^s (l+r_u)\mathrm{d}u}\mathrm{d}s\right)}_{C_1} + \underbrace{\Big(1-\rho\Big)E\left(e^{-\int_0^{\tau_1}(l+r_u)\mathrm{d}u}S_{\tau_1}\right)}_{C_2}.
\end{aligned}$$

As with the proof for B_1,

$$\begin{aligned} C_1 &= D_3(c_3+l)\Bigg(Y_0'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde H(0)\Big(Q-\mathrm{diag}(\hat r+l)\Big)^{-1}\mathbf{1} \\ &\quad -J_0'\Big(Q-\mathrm{diag}(\hat r+l)\Big)^{-1}\mathbf{1}\Bigg). \end{aligned}$$

Then, similar to the proof for S_3, S_4,

$$\begin{aligned} C_2 &= (1-\rho)\Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big)Y_0'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde H(0)\Big(Q-\mathrm{diag}(\hat r)\Big)^{-1}\mathbf{1} \\ &\quad -(1-\rho)\delta\alpha\Big(D_1+D_2+D_3\Big)Y_0'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde H(1)\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat r\big)\Big)^{-1}\mathbf{1} \\ &\quad +(1-\rho)\pi_1V_0Y_0'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde K(1)W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde I \\ &\quad +(1-\rho)\Big(1-\gamma\Big)\Big(c_1D_1+c_2D_2\Big)Y_0'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde K(0) \\ &\quad W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde H(0)\big(Q-\mathrm{diag}(\hat r)\big)^{-1}\mathbf{1} \\ &\quad -(1-\rho)\delta V_0{Y_0}'W^-e^{Q^{(\hat r+l,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde K(1) \\ &\quad W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde H(1)\Big(Q+\mathrm{diag}\big(\varphi(1)-\hat r\big)\Big)^{-1}\mathbf{1}. \end{aligned}$$

4.5 Bank Value

The total bank value is

$$\begin{aligned} v &= V_0+\gamma E\left(\int_0^{\tau_1}e^{-\int_0^s r_u\mathrm{d}u}c_3D_3\mathrm{d}s+\int_0^{\tau_2}e^{-\int_0^s r_u\mathrm{d}u}\Big(c_1D_1+c_2D_2\Big)\mathrm{d}s\right)-DI \\ &\quad -E\left(e^{-\int_0^{\tau_2}r_u\mathrm{d}u}\Big(1-\pi_1-\pi_2\Big)V_{\tau_2}\right) \end{aligned}.$$

so that

$$\begin{aligned} v &= V_0+\gamma c_3D_3\left(Y_0'W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2+D_3)}{V_0}\right)}\tilde H(0)\Big(Q-\mathrm{diag}(\hat r)\Big)^{-1}\mathbf{1}-J_0'\Big(Q-\mathrm{diag}(\hat r)\Big)^{-1}\mathbf{1}\right) \\ &\quad +\gamma(c_1D_1+c_2D_2)\left(Y_0'W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde H(0)\Big(Q-\mathrm{diag}(\hat r)\Big)^{-1}\mathbf{1}-J_0'\Big(Q-\mathrm{diag}(\hat r)\Big)^{-1}\mathbf{1}\right) \\ &\quad -DI-(1-\pi_1-\pi_2)V_0Y_0'W^-e^{Q^{(r,-)}\left(x-\ln\frac{\alpha(D_1+D_2)}{V_0}\right)}\tilde I. \end{aligned}$$

4.6 Conversion and Default Probabilities

From proposition 4, we obtain:

$$E\left(e^{-\kappa_1\tau_1}\right) = Y_0' W^- e^{Q^{(\hat{\kappa}_1,-)}\left(x - \ln \frac{\alpha(D_1+D_2+D_3)}{V_0}\right)} \mathbf{1}$$

and

$$E\left(e^{-\kappa_2\tau_2}\right) = Y_0' W^- e^{Q^{(\hat{\kappa}_2,-)}\left(x - \ln \frac{\alpha(D_1+D_2)}{V_0}\right)} \mathbf{1}$$

where $\mathbf{1} \in \mathbb{R}^{2n}$ is a vector of ones, $\hat{\kappa}_1 = (\kappa_1, ..., \kappa_1)'$ and $\hat{\kappa}_2 = (\kappa_2, ..., \kappa_2)'$. Denote $f_1(t) = P(\tau_1 \leq t)$ and $f_2(t) = P(\tau_2 \leq t)$. Their Laplace transforms satisfy:

$$\hat{f}_1(\kappa_1) = \int_0^\infty e^{-\kappa_1 t} P(\tau_1 \leq t) \mathrm{d}t = \frac{1}{\kappa_1} \int_0^\infty e^{-\kappa_1 t} dP(\tau_1 \leq t) = \frac{1}{\kappa_1} E\left(e^{-\kappa_1\tau_1}\right)$$

and

$$\hat{f}_2(\kappa_2) = \int_0^\infty e^{-\kappa_2 t} P(\tau_2 \leq t) \mathrm{d}t = \frac{1}{\kappa_2} \int_0^\infty e^{-\kappa_2 t} dP(\tau_2 \leq t) = \frac{1}{\kappa_2} E\left(e^{-\kappa_2\tau_2}\right).$$

We can compute the conversion and default probabilities $P(\tau_1 \leq t)$ and $P(\tau_2 \leq t)$ by numerically inverting the above Laplace transforms.

5 Numerical Illustration

We compare the conversion and default probabilities for the regime switching Brownian motion model (RSBM) and the regime switching jump diffusion model (RSJD). We choose the parameters to make the first and second order moments equal in the two models. We use Proposition 4 to obtain the Laplace transform of τ and we use the Gaver-Stephest algorithm to perform the numerical Laplace inversion for computing the conversion and default probabilities.

Table 1: The parameters for the two states RSBM

Parameter	Low-volatility	High-volatility
μ	0.07	0.05
σ	0.2230	0.4500

For this illustration, we assume the following transition matrix:

$$Q = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}$$

and we set the various parameters as in Tables 1, 2, and 3.

The left panel of Figure 1 shows the term structure of the conversion probability when the bank asset returns switch between diffusions or between jump-diffusions. It appears that the conversion probability is higher in the presence of jumps.

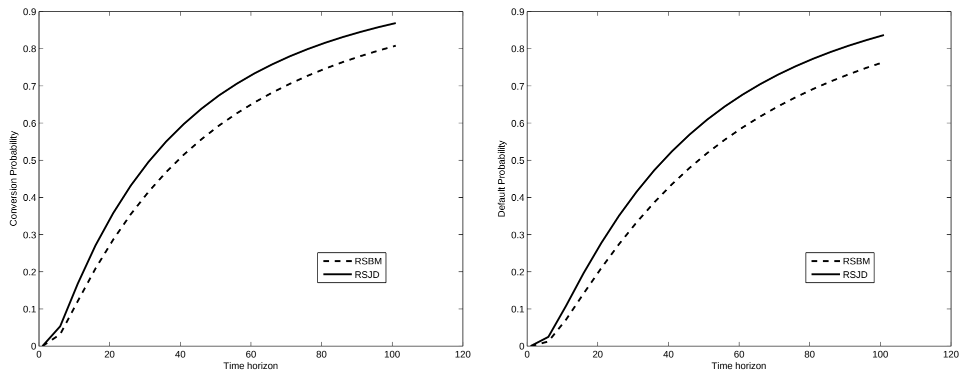
Table 2: The parameters for the two states RSJD

Parameter	Low-volatility	High-volatility
μ	0.07	0.05
σ	0.2	0.4
λ	15	25
η_1	1/0.02	1/0.03
η_2	1/0.03	1/0.05
p	0.5	0.5

Table 3: Other parameters

r	0.025
D_1	25
D_2	25
D_3	25
V_0	100
α	0.1
J_0	(0,1)’
Y_0	(0,1,0,0,0,0)’

The right panel of Figure 1 shows the term structure of the default probability when the bank asset returns switch between diffusions or between jump-diffusions. It appears that the default probability is higher in the presence of jumps.



(a) (b)

Figure 1: (a) Conversion Probability (b) Default Probability

References

Barlow, M. T., L. Rogers, and D. Williams (1980): “Wiener-Hopf factorization for matrices,” *Séminaire de Probabilités de Strasbourg*, 14, 324–331.

Black, F., and J. C. Cox (1976): “Valuing corporate securities: Some effects of bond indenture provisions,” *The Journal of Finance*, 31(2), 351–367.

Chen, N., and S. G. Kou (2009): "Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk," *Mathematical Finance*, 19(3), 343–378.

Elliott, R. J., L. Aggoun, and J. B. Moore (2008): *Hidden Markov models: estimation and control*, vol. 29. Springer Science & Business Media.

Elliott, R. J., L. Chan, and T. K. Siu (2005): "Option pricing and Esscher transform under regime switching," *Annals of Finance*, 1(4), 423–432.

Elliott, R. J., and C.-J. U. Osakwe (2006): "Option pricing for pure jump processes with Markov switching compensators," *Finance and Stochastics*, 10(2), 250–275.

Elliott, R. J., T. K. Siu, L. Chan, and J. W. Lau (2007): "Pricing options under a generalized Markov-modulated jump-diffusion model," *Stochastic Analysis and Applications*, 25(4), 821–843.

Glasserman, P., and B. Nouri (2012): "Contingent capital with a capital-ratio trigger," *Management Science*, 58(10), 1816–1833.

Hainaut, D., Y. Shen, and Y. Zeng (2016): "How do capital structure and economic regime affect fair prices of banks equity and liabilities?," *Annals of Operations Research*, pp. 1–27.

Hamilton, J. D. (1989): "A new approach to the economic analysis of nonstationary time series and the business cycle," *Econometrica: Journal of the Econometric Society*, pp. 357–384.

Hilberink, B., and L. C. G. Rogers (2002): "Optimal capital structure and endogenous default," *Finance and Stochastics*, 6(2), 237–263.

Jiang, Z., and M. R. Pistorius (2008): "On perpetual American put valuation and first-passage in a regime-switching model with jumps," *Finance and Stochastics*, 12(3), 331–355.

Le Courtois, O., and F. Quittard-Pinon (2006): "Risk-neutral and actual default probabilities with an endogenous bankruptcy jump-diffusion model," *Asia-Pacific Financial Markets*, 13(1), 11–39.

Leland, H. E. (1994a): "Bond prices, yield spreads, and optimal capital structure with default risk," *Working paper, University of California at Berkeley.*

——— (1994b): "Corporate debt value, bond covenants, and optimal capital structure," *The Journal of Finance*, 49(4), 1213–1252.

Leland, H. E., and K. B. Toft (1996): "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads," *The Journal of Finance*, 51(3), 987–1019.

Lin, S.-K., Y.-M. Lian, and S.-L. Liao (2014): "Pricing gold options under Markov-modulated jump-diffusion processes," *Applied Financial Economics*, 24(12), 825–836.

Longstaff, F. A., and E. S. Schwartz (1995): "A simple approach to valuing risky fixed and floating rate debt," *The Journal of Finance*, 50(3), 789–819.

Merton, R. C. (1974): "On the pricing of corporate debt: The risk structure of interest rates," *The Journal of Finance*, 29(2), 449–470.

Rogers, L. C. G., and Z. Shi (1994): "Computing the invariant law of a fluid model," *Journal of Applied Probability*, 31(4), 885–896.