Risk Control of Variable Annuities With Ratchet

Abstract

This paper suggests a unified methodology for the management of Guaranteed Minimum Accumulation Benefit contracts. Using a non-Gaussian setting in line with many of the stylized features observed in the market, we address the pricing, hedging, and risk control of these contracts from an operational risk management perspective. Since the well-known and widely used delta-hedging ratio is not optimal, one of the most important problems raised is the issue of hedging. The literature suggests many theoretical solutions whose efficiency from a computational point of view is controversial and rarely studied. From the empirical part of the paper, the authors give a simple rule for designing a hedging policy appropriate to the actual financial environment that proves useful both for insurers and regulators.

JEL: G22, G11
Keywords: Variable Annuities, GMAB, Ratchet, Pricing, Hedging, Lévy processes.

Introduction

Variable Annuities, hereafter VA, are life insurance contracts linked to financial markets in such a way that policyholders or their beneficiaries can take advantage of rising markets whilst being protected if prices plummet. Examples of such contracts are Guaranteed Minimum Maturity Benefit (GMMB) and Guaranteed Minimum Death Benefit (GMDB) contracts. In this article, we consider the Guaranteed Minimum Accumulation Benefit (GMAB) contract with a ratchet feature, which is more general and encompasses the two previous contracts. Since it is well recognized that financial prices present jumps, that their return distributions have fat tails, and that option prices exhibit smile effects, traditional geometric Brownian motion is not appropriate for modeling financial asset prices. We therefore model these prices using exponential of Lévy processes. More precisely, we consider Merton- and Kou-type jump diffusions, and Lévy processes with infinite activity such as variance gamma and CGMY processes. The literature on the pricing of VA and Equity Indexed Annuities (EIA) is abundant. We note, for example, the general framework developed by Bacinello (2003) and the references therein or the recent paper by Gerber et al. (2013). The hedging issue is considered less often and is rarely studied from an operational point of view. The hedging problem is difficult because of jumps. As a consequence, the market is incomplete and no perfect duplication can be obtained. In this paper, we use a quadratic hedging approach following the analysis in Cont and Tankov (2004) and that in Boyarchenko and Levendorskiĭ (2000). The GMAB studied here is presented in Hardy (2003). This contract has a sophisticated protection design, which makes it difficult to price and hedge, in particular because the guarantee can be changed at certain reset dates. The new guarantee depends on the evolution of policyholder account value and can induce upwards jumps in this value. This mechanism generates a ratchet effect and offers a dynamic fund protection. An example of a ratchet contract is given in Tanskanen and Lukkarinen (2003). Here, we employ a Fourier analysis in line with Boyarchenko and Levendorskiĭ (2000). One of the first papers to use Fourier analysis for option pricing was Carr and Madan (1998). It is now common practice and many articles and books have been published on this topic see Cherubini et al. (2010). We chose the framework proposed by Boyarchenko and Levendorskiĭ (2000) because it provides a very efficient method for both pricing and hedging, involving the computation of an integral easily calculated by a Fast Fourier Transform algorithm.

This paper is organized as follows. In section 1, we describe the GMAB contract, the assumptions made, and the method of valuation chosen to obtain fair costs. In section 2, we specify the technique used for pricing and hedging in a Lévy market as well as the stochastic processes used. Section 3 is devoted to our numerical illustration, while section 4 concludes the paper.

1 Product and Modeling

Let T be the expiration date of the contract, assumed issued at time 0, and let T_x be the residual life of a policyholder aged x at the issue date. The contract payoff is of the following type:

$$\max(A_{T\wedge T_x}, G_{T\wedge T_x}),$$

where $A_{T\wedge T_x}$ is the policyholder account value at time $T\wedge T_x$ and $G_{T\wedge T_x}$ is the guarantee at that time. Because

$$\max(A_{T\wedge T_x}, G_{T\wedge T_x}) = A_{T\wedge T_x} + [G_{T\wedge T_x} - A_{T\wedge T_x}]^+,$$

the contract can be considered, from the policyholder's point of view, as a long position in the fund and a put option with the strike price of $G_{T\wedge T_x}$. The insurer is at risk because of the short position in the option. This embedded option, sometimes known as an optional rider, is paid continuously rather than upfront. Let m denote the offset ratio necessary for financing the guarantee liability. The fees are continuously deducted at the rate m from the reference equity portfolio in which the policyholder's single upfront premium has been invested. We denote by S the reference portfolio price process, and by ${}_tp_x = \Pr[T_x > t]$ the conditional survival probability under the physical measure $\mathcal{P}$. The probability of dying ${}_tq_x$ is $1 - {}_tp_x$. When $[0, T]$ is partitioned into subperiods (years or months, for example), the fees[1] are paid at the beginning

[1]In fact the fees are used by the insurer to pay expenses other than the guarantee liability. For the sake of simplicity, we only consider the offset ratio.

of each subperiod. If death occurs during a subperiod, the benefit is assumed to be paid at the end of this subperiod. A Guaranteed Minimum Accumulation Benefit contract offers the possibility of modifying the guarantee at certain specified dates, thus adding a path-dependent feature. This ratchet feature affects the dynamics of policyholder account value, A. More precisely, consider the following sequence $\mathcal{T}$ of rollover dates:

$$\mathcal{T} = \{t_1, t_1 < t_2, ..., t_{n-1} < t_n = T\},$$

enabling the guarantee to be modified at each of these times in the following way: let A_t^- and A_t^+ be the account values just before and just after the guarantee is reset at time t. Note that at each time $t_i \in \mathcal{T}$, the fund value immediately before renewal, $A_{t_i}^-$, is related to the fund value brought forward from time t_{i-1}, $A_{t_{i-1}}^+$, through the relationship:

$$A_{t_i}^- = A_{t_{i-1}}^+ \frac{S_{t_i}}{S_{t_{i-1}}} e^{-m(t_i - t_{i-1})}. \tag{1}$$

Immediately before each reset date t_i, the insurer compares the account value $A_{t_i}^-$ to $G_{t_{i-1}}$. Given that the contract is still in force, two scenarios are possible:

. If $A_{t_i}^- < G_{t_{i-i}}$, the insurer pays the difference $G_{t_{i-1}} - A_{t_i}^-$ into the policyholder's account, so immediately after t_i, $A_{t_i}^+ = G_{t_{i-1}}$, then G_{t_i} is worth $A_{t_i}^+$. Note that $A_{t_i}^+ > A_{t_i}^-$.

. If $A_{t_i}^- > G_{t_{i-1}}$, there is no cash-in for the insurer, but G_{t_i} is reset to $A_{t_i}^-$ and immediately after, $A_{t_i}^+$ is worth G_{t_i}. Note that $A_{t_i}^+ = A_{t_i}^-$.

In an equivalent way, at each t_i,

$$G_{t_i} = A_{t_i}^+ = \max\left(G_{t_{i-1}}, A_{t_i}^-\right) = A_{t_i}^- + L_{t_i},$$

where,

$$L_{t_i} := [G_{t_{i-1}} - A_{t_i}^-]^+, \tag{2}$$

is the guarantee option payoff at time t_i. The starting guarantee, G_0, could be part of the initial premium and the starting account value A_0 is assumed to be worth S_0. Between reset dates, the account value A is linked to the reference portfolio value S according to:

$$A_t = A_{t_{i-1}}^+ \frac{S_t}{S_{t_{i-1}}} e^{-m(t - t_{i-1})} \quad t_{i-1} \le t < t_i \;\; \forall t_{i-1}\,, t_i \in \mathcal{T}. \tag{3}$$

Note that this mechanism affects the account value dynamics. Because $A_{t_i}^-$ can be different from $A_{t_i}^+$ at reset dates, t_i, $\forall t_i \in \mathcal{T}$, the policyholder account value has possible upward jumps at these dates. These jumps occur because of the GMAB design and are different from those coming from the market (possible jumps in the S process). Figure 1 illustrates the mechanism of this contract for a sample path of the reference portfolio. In this example, we consider three reset dates at time 5, 10, and 15. We observe that policyholder account value jumps at the first reset date, while the guarantee jumps at the second. In this simulation, we also note that the account value is always equal to or greater than the reference portfolio value, which makes this guarantee very attractive for the policyholder. The first problem to be solved is to find the fair fees value. This is addressed in the following subsection.

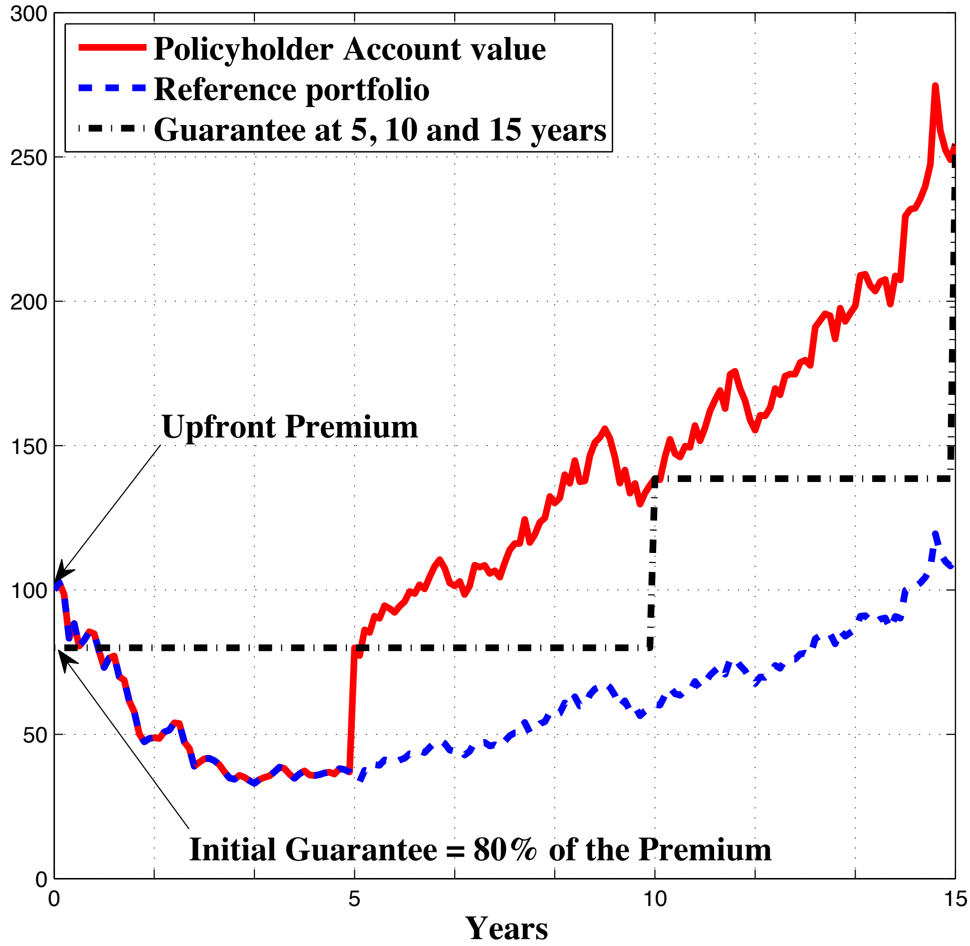


Figure 1: Policyholder account value process A, guarantee G and reference portfolio S, for a GMAB with ratchet at reset dates 5, 10, and 15 years: $m = 3\%$.

1.1 Fair Market Value

The contract analyzed in this article can be considered as a contingent claim in a combined financial-insurance market. Formally, using standard notations, $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P}_1\right)$ is a filtered probability space associated with the financial market. The probability measure $\mathcal{P}_1$ can be the historical or real measure $\mathcal{P}$ or a risk-neutral probability measure Q. We use underscripts to refer to a particular measure and omit the reference if it is not necessary to specify the measure or if it is obvious from the context. As we focus on market risk, we assume that mortality risk can be diversified away and hence $\mathcal{P}$ and Q coincide on mortality-related events. We also assume that financial risk and mortality risk are independent under Q, see Dhaene et al. (2013) for interesting developments on this subject. We use a superscript in expectation, variance, or covariance operators to refer to a conditional expression given the information available. For example $E^t_Q[A]$ is the conditional expectation of A under the risk-neutral measure Q given $\mathcal{F}_t$.

Let r be the instantaneous interest rate in the given economy and δ_t, the discount factor at time t, defined as

$$\delta_t := \exp\left\{-\int_0^t r_u du\right\}.$$

We assume that the reference equity portfolio value follows a geometric Lévy process under a risk-neutral probability measure Q. Thus,

$$\frac{dS_t}{S_{t-}} = r_t dt + dM_t, \tag{4}$$

where M is a martingale that will be defined precisely below. We denote by $M\&E$ (mortality and expense) the expected discounted value of all the fees paid until the contract is in force, i.e., until death or maturity, whichever comes first. The $M\&E$ associated with the contract has the following market value:

$$M\&E(m) = A_0\Big\{{}_Tp_x(1-e^{-mT}) + \sum_{t=0}^{T-1} {}_tp_x q_{x+t}(1-e^{-m(t+1)})\Big\}. \tag{5}$$

We denote by $\xi_{S_0,m,T}$ the initial price of the optional rider in the GMAB with expiry T. Following Milevsky and Posner (2001) and Quittard-Pinon and Randrianarivony (2011), the equilibrium value or the fair price for the guarantee is thus the solution in m of the equation

$$\xi_{S_0,m,T} = M\&E(m). \tag{6}$$

Put differently, the fair cost is such that the value associated with the discounted continuous cash flows coming from the fees is equal to the contract's optional rider value. It is worth noting that the solution of equation (6) depends on the choice of a risk-neutral measure, and there is therefore no unique solution for m. Hence, a risk-neutral measure must be chosen. We consider this choice in our discussion of the Q-measure in section 2.

1.2 Valuation: General Formulae

To price the guarantee, we use the arbitrage pricing theory in continuous time. We assume a constant interest rate r, and denote by $P(S_0, K, \tau)$ the price of a European put option, written on S, with an initial value equal to S_0, an exercise price of K, and a maturity τ. As described in section 1, the GMAB includes a path-dependent feature at each rollover date in the sequence $\mathcal{T}$, allowing the insured to accumulate the maximum of either the current policyholder account value or the maximum value recorded at the previous reset dates. Although the valuation of this product is not straightforward, general formulae can nevertheless be obtained. Let $H_i(S_0, m, t_i)$ be the price at time 0 of the i^{th} guarantee option,

$$H_i(S_0, m, t_i) := E_Q[\delta_{t_i} L_{t_i}].$$

We have the following result:

Proposition 1. *For all $k > 1$, the initial price of the k^{th} guarantee option at $t_k \in \mathcal{T}$ is obtained via the following recursive formula:*

$$H_k(S_0, m, t_k) = \Big(S_0 e^{-mt_{k-1}} + \sum_{i=1}^{k-1} H_i(S_0, m, t_i) e^{-m(t_{k-1}-t_i)} \Big) \\ \times P(e^{-m(t_k - t_{k-1})}, 1, t_k - t_{k-1}), \quad k > 1, \tag{7}$$

where the starting step is

$$H_1(S_0, m, t_1) = P(S_0 e^{-mt_1}, G_0, t_1).$$

We now introduce mortality. The initial value, $\xi_{S_0,m,\mathcal{T}}$, of the insurer's liability, i.e the GMAB optional rider, is

$$\xi_{S_0,m,\mathcal{T}} = \sum_{i=1}^{\#\mathcal{T}} \xi_{S_0,m,t_i}, \tag{8}$$

where $\#\mathcal{T}$ is the cardinality of $\mathcal{T}$ and ξ_{S_0,m,t_k}, is given by:

$$\xi_{S_0,m,t_k} = \Big(S_0 e^{-mt_{k-1}} + \sum_{i=1}^{k-1} H_i(S_0, m, t_i) e^{-m(t_{k-1}-t_i)} \Big) \\ \times \Bigg\{ \sum_{t=t_{k-1}}^{t_k - 1} {}_t p_x q_{x+t} P\big(e^{-m(t - t_{k-1}+1)}, 1, t - t_{k-1} + 1\big) \\ + {}_{t_k} p_x P\big(e^{-m(t_k - t_{k-1})}, 1, t_k - t_{k-1}\big) \Bigg\}, \quad \forall\, k > 1,$$

with

$$\xi_{S_0,m,t_1} = \sum_{t=0}^{t_1 - 1} {}_t p_x q_{x+t} \times H_1(S_0, m, t+1) + {}_{t_1} p_x \times H_1(S_0, m, t_1).$$

Some explanatory remarks are in order. It should be noted that ξ_{S_0,m,t_i} and H_i are different objects. They are building blocks for obtaining the GMAB optional rider price formula (8). The result in Proposition 1 does not depend on any particular dynamics of the financial prices. All that is needed is the independence of non-overlapping increments of the process X in (9), which is, by definition, the case for the increments of a Lévy process. It is worth noting that only the sum in brackets of (7) depends on the current stock price. This observation is useful for assessing the hedge portfolio. Also note that the GMAB benefit shown in (8) is reduced to a mixed GMMB/GMDB, provided that there is no reset date. This GMAB is therefore a path-dependent generalization of GMMB/GMDB contracts. The price of the optional rider of the contract analyzed in this article is given in (8) as the sum of non-straightforward combinations of European put option prices. To make it operational, we need to determine how to model the financial prices and mortality. This is developped in the next section.

2 Pricing and Hedging

In this section, we list the five Lévy models used in this paper : arithmetic Brownian motion, Merton and Kou jump diffusions, variance gamma, and CGMY processes. We employ the Fourier method and the Boyarchenko and Levendorskiĭ (2000) approach in particular. For pricing, we note their formula, and for hedging we follow their suggestion and compare their hedging ratio to the one obtained by Cont and Tankov (2004) in the jump diffusion. We prove that these ratios coincide. We also discuss the choice of the Q measure and justify the choice of a particular Esscher measure. We end this section by considering hedging in practice.

As assumed in section 1.1, the price of the equity portfolio is modeled by the exponential of a Lévy process X:

$$S_t = S_0 e^{X_t}. \tag{9}$$

Let us denote by ϕ_t the characteristic function of X at time t. A Lévy process can be completely specified by its characteristic exponent, ψ in

$$\phi_t(u) = E[e^{iuX_t}] = e^{-t\psi(u)}, \quad t \geq 0, \tag{10}$$

given by the Lévy-Khintchine formula

$$\psi(u) = -iu\mu + \frac{1}{2}\sigma^2 u^2 - \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux1_{|x|<1}\right) \nu(x)dx. \tag{11}$$

The triplet (μ, σ, ν) fully specifies X and is referred to as the Lévy characteristics. In the light of (10), a Lévy process can generate a wide range of characteristic exponent behaviors through a flexible specification of the Lévy density $\nu(x)$. The sample paths of a pure jump Lévy process exhibit finite activity when the integral of the Lévy density is finite: $\int_{-\infty}^{+\infty} \nu(x)dx = \lambda < \infty$ where λ measures the mean arrival rate of jumps. A finite[2] activity jump process generates a finite number of jumps within any finite time interval.

2.1 Examination of Lévy Processes

The only continuous Lévy process is the arithmetic Brownian motion. Geometric Brownian motion is the basic model in continuous time finance, see Black and Scholes (1973), and is obtained with $\nu(x) = 0$ in equation (11) for all x. The characteristic exponent is then:

$$\psi^{BS}(u) = -iu\mu + \frac{1}{2}\sigma^2 u^2.$$

Merton (1976) incorporates an additional compound Poisson process with mean arrival rate λ, so the characteristic exponent is

$$\psi^M(u) = -iu\mu + \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{i\gamma u - \frac{1}{2}\delta^2 u^2} - 1\right).$$

[2] When the integral is infinite, the sample paths exhibit infinite activity, and generate an infinite number of jumps within any finite interval.

The random jump size, conditional on occurrence of one jump, is normally distributed with mean γ and variance δ^2. Using the compound Poisson jump-type process, Kou (2002) suggests an asymmetric double exponential distribution for the random jump sizes. The characteristic exponent is:

$$\psi^K(u) = -iu\gamma + \frac{1}{2}\sigma^2 u^2 + \lambda\Big(\frac{p\lambda_1}{\lambda_1 - iu} + \frac{q\lambda_2}{\lambda_2 + iu} - 1\Big), \quad \lambda_1, \lambda_2 > 0,$$

with $p \geq 0$ and $q \geq 0$ constrained by $p + q = 1$. Also, u has to be such that Im u is in $(-\lambda_1, \lambda_2)$. Although it is appropriate to use compound Poisson jumps to capture non-negligible probabilities of rare and large events such as market crashes (as illustrated by Figures 2a and 2b), many authors observe that asset prices actually display many small jumps. These types of features are better explained by infinite-activity jumps. A popular example is the variance gamma (VG) model introduced by Madan and Seneta (1990), which is obtained by time changing a Brownian motion with a gamma subordinator whose one unit-time increment variance rate is ς. The characteristic exponent is

$$\psi^{VG}(u) = -iu\mu + \frac{1}{\varsigma}\ln(1 - iu\theta\varsigma + u^2\frac{\sigma^2}{2}\varsigma), \quad \varsigma > 0, \ \ \theta \in \mathbb{R},$$

with the regularity strip

$$-\theta - \sqrt{\theta^2 + \frac{2\sigma^2}{\varsigma}} < \sigma^2(\text{Re u}) < -\theta + \sqrt{\theta^2 + \frac{2\sigma^2}{\varsigma}}.$$

Figure 2c shows the deformation of the sample path distribution as a function of parameter θ of the VG. Another popular example that can generate different jump types is the CGMY model of Carr et al. (2002), with the following characteristic exponent:

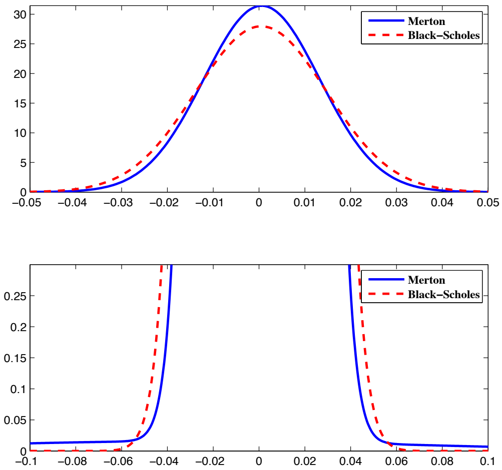
$$\psi^{CGMY}(u) = -C\,\Gamma(-Y)\left[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right],$$

where $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$. This kind of process is a generalization of the variance gamma Lévy density ($Y = 0$), with parameters identified by Carr et al. (2002). The CGMY parameters play an important role in capturing certain properties of the process sample paths. The parameter C describes the intensity of the process; it plays a part similar to that of the variance of the Brownian motion. The parameters G and M respectively control the rate of exponential decay on the right and on the left tails of the Lévy density, leading to a skew when they are unequal. See Figure 2d for an illustration of this point.

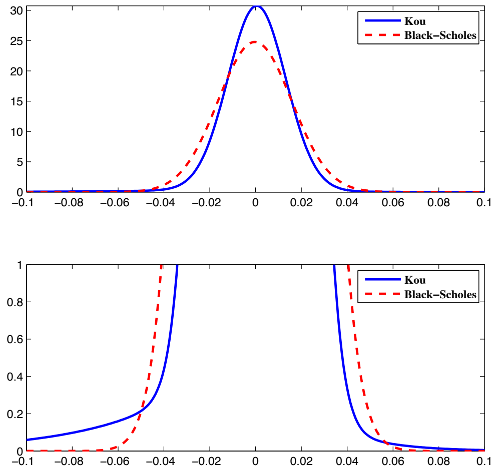
2.2 Pricing

Using a generalized Fourier approach, and following Boyarchenko and Levendorskiĭ (2000), the European put option defined in section 1.2 can be expressed as

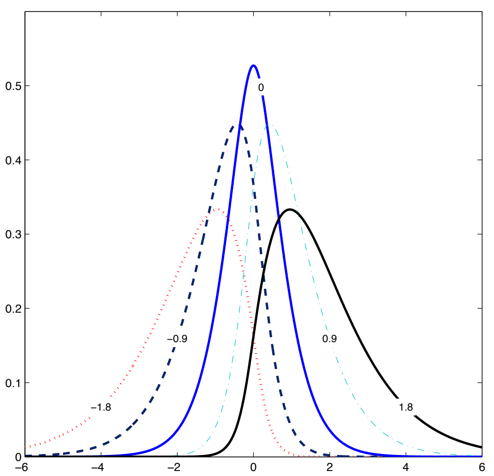
$$P(S_t, K, \tau) = K\frac{1}{2\pi}e^{-bx'}\int_{\mathbb{R}} e^{iux'}\frac{e^{-\tau(r+\psi(u+ib))}}{(-iu+b)(-iu+b+1)}du, \qquad (12)$$



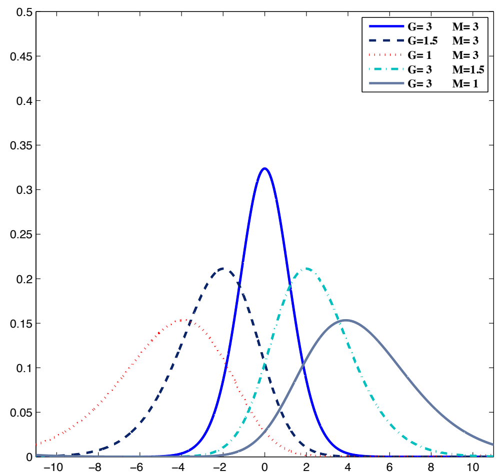
(a) Comparison of the overall shapes of densities under the Merton and Black-Scholes models with the same mean and variance. Parameters used are $T = 1/250$, $\lambda = 1$, $\mu = 0.15$, $\sigma = 0.2$, $\gamma = -0.03$, and $\delta = 0.1$.



(b) Comparison of the overall shapes of densities under the Kou and Black-Scholes models with the same mean and variance. Parameters used are $T = 1/250$, $\lambda = 10$, $\mu = 0.15$, $\sigma = 0.20$, $\lambda_1 = 1/0.02$, $\lambda_2 = 1/0.04$, and $p = 0.3$.



(c) VG densities for various values of θ. To replicate the figure, use $T = 1$, $\varsigma = 0.4$, $\sigma = 0.9$, and $\mu = 0$.



(d) CGMY densities for various values of parameters G and M. To replicate, use $T = 1$, $C = 7$, and $Y = 0.1$.

Figure 2: Densities of some Lévy processes obtained via Fourier inversion.

with $b > 0$, $x' = \ln(S_t/K)$, $\tau = T - t$. The Equivalent Martingale Measure condition (EMM) ensures that discounted prices are Q-martingales:

$$r + \psi(-i) = 0.$$

This last relationship can be used to express μ via the other parameters of the Lévy process:

$$\mu = r - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} \left(1 - e^x + x 1_{|x|<1} \nu(dx)\right).$$

Note that formula (12) holds for a European call option with $b < -1$. Taking into account the above EMM condition, this formula can be obtained through a quadrature easily, quickly, and accurately using FFT. In our experience, a practical choice for parameter b is -3 and 3, for a call and for a put-like option, respectively.

2.3 Hedging

With the exception of the Gaussian case, Lévy processes present jumps. If financial prices are modeled with jump processes, the market is incomplete. The main difficulty is therefore that hedging becomes an approximation problem and the usual delta ratio does not result in a strategy that perfectly replicates the option. In this section and in the following (2.3.1) we omit the reference to the probability measures, Q and $\mathcal{P}$, when computing expectations and variances, only precisely defining these measures when necessary. Cont and Tankov (2004) choose the strategy that minimizes the mean square terminal hedging error. They denote by $F(S,t)$ the option price as a function of its state variables S and t, and obtain the following optimal ratio Λ_t:

$$\Lambda_t = \frac{\sigma^2 \frac{\partial F}{\partial S}(S_t, t) + \frac{1}{S} \int_{-\infty}^{+\infty} [F(S_t e^x, t) - F(S_t, t)](e^x - 1)\nu(dx)}{\sigma^2 + \int_{-\infty}^{+\infty} (e^x - 1)^2 \nu(dx)}. \tag{13}$$

The Lévy measure ν can differ in the historical and risk-neutral worlds. The numerator of formula (13) exhibits a sum of two terms: the sensitivity of the derivative price $F(S,t)$ to infinitesimal movements of the stock price S, and the average sensitivity to finite-sized jumps. Note that the implementation of this formula is not straightforward because of the integral term $\int_{-\infty}^{+\infty} F(t, S_t e^x)\nu(dx)$, which depends on the whole solution $F(t, .)$. The presence of this term necessitates the use of many quadratures or the non-simple resolution of PIDE schemes. So far, equation (13) can be quasi-explicitly computed in the MJD economy, see section 2.3.3. We now briefly recall the solution for hedging suggested by Boyarchenko and Levendorskiĭ (2000). For this purpose, consider an investor with a sufficient wealth W_t at time t who takes a short position in a European option with the expiration date T, a long position in Θ_t units of the underlying, and who invests the residual in the money market account. The next step is to determine the optimal hedging ratio Θ that minimizes the variance of the portfolio during the next small time interval. In other words, if we denote this residual by $w_0(t)$, the investor's wealth at time $t + \Delta t < T$ will be given by:

$$W_{t+\Delta t} = -F(S_{t+\Delta t}, t + \Delta t) + \Theta_t S_{t+\Delta t} + e^{r\Delta t} w_0(t).$$

Then the optimal allocation Θ is such that the conditional variance of wealth is (locally) minimized, i.e.,

$$\inf_{\Theta} E^t\left[\left(W_{t+\Delta t} - E^t[W_{t+\Delta t}]\right)^2\right]. \tag{14}$$

The solution is expressed in the following proposition:

Proposition 2. *In the historical world the optimal ratio (14) is:*

$$\Theta'(S_t, T) = K\frac{1}{2S_t\pi}e^{-bx'}\int_{\mathbb{R}} e^{iux'}\frac{e^{-\tau(r+\psi_Q(u+ib))}B_{\mathcal{P}}(u+ib)}{(-iu+b)(-iu+b+1)}du, \tag{15}$$

with $b > 0$, $x' = \ln(S_t/K)$ and with

$$B_{\mathcal{P}}(u) = \frac{-\psi_{\mathcal{P}}(u-i) + \psi_{\mathcal{P}}(u) + \psi_{\mathcal{P}}(-i)}{-\psi_{\mathcal{P}}(-2i) + 2\psi_{\mathcal{P}}(-i)}.$$

In the risk-neutral universe, it is worth

$$\Theta(S_t, T) = K\frac{1}{2S_t\pi}e^{-bx'}\int_{\mathbb{R}} e^{iux'}\frac{e^{-\tau(r+\psi_Q(u+ib))}B_Q(u+ib)}{(-iu+b)(-iu+b+1)}du, \tag{16}$$

and

$$B_Q(u) = \frac{-\psi_Q(u-i) + \psi_Q(u) + \psi_Q(-i)}{-\psi_Q(-2i) + 2\psi_Q(-i)}.$$

Proof. See Boyarchenko and Levendorskiĭ (2000). A detailed proof can be obtained from the authors upon request. □

As in section 2.2, formula (16) also remains valid for a European call-like option with $b < -1$. The above hedging ratio is therefore obtained in a similar way to the pricing formula in (12) up to factors $B(.)$ and S_t^{-1}, thereby unifying the pricing and the hedging methods.

2.3.1 Θ and Λ ratios

Although the Θ and Λ ratios have distinct expressions that can lead to different types of computations, a property we exploit in the next subsection, we show here that they are identical and justify (13) using the simple criterion (17). As in Black and Scholes (1973),we consider a self-financing portfolio consisting of a long position in the derivative and a short position of θ_t units in the underlying asset. We choose the optimal quantity to invest in the underlying such that the risk, measured by the variance of variation on the portfolio value during a short period, $(t, t+\Delta t]$, is minimized. Note that in the Black and Scholes model this risk is zero, which is not the case here, because we work in an incomplete market. Denote:

$$V_t := F_t - \theta_t S_t.$$

With obvious notations,

$$\Delta V = \Delta F - \theta_t \Delta S.$$

We need to solve the optimization problem

$$\inf_{\theta_t} \left[Var^t[\Delta F - \theta_t \Delta S] \right], \tag{17}$$

which is exactly (14), leading to (15) in the real-world and (16) in the risk-neutral one. Now consider another way of computing $Var^t[\Delta V]$. We begin by noting that

$$\Delta V = \int_t^{t+\Delta t} dV = \int_t^{t+\Delta t} (dF - \theta_t dS_t).$$

To go farther, we need to specify the dynamics of the risky asset price in (9). To do that we introduce the Poisson jump measures $J_X(dy, dt)$ associated with the processes X, and its compensated Poisson jump measures $\tilde{J}_X$:

$$\tilde{J}_X(dy, dt) = J_X(dy, dt) - \nu(dx)dt,$$

Using Itō's lemma the dynamics of the risky asset can be written:

$$\frac{dS}{S_{t-}} = \mu dt + \sigma dW_t + \int_{\mathbb{R}} (e^x - 1)\tilde{J}_X(dx, dt). \tag{18}$$

Where μ is the dift term, note that $\mu = r$ in the risk-neutral world. Now, by again applying Itō's lemma to F and observing that the drift does not intervene in the conditional variance, we obtain

$$Var^t[V_{t+\Delta t} - V_t] = Var^t[A + B],$$

where

$$A := \int_t^{t+\Delta t} \left(\frac{\partial F}{\partial S}(S_{u-}, u) - \theta_u \right) \sigma S_u dW_u$$

$$B := \int_t^{t+\Delta t} \int_{\mathbb{R}} (F(S_{u-}e^x, u) - F(S_{u-}, u) - \theta_u S_u (e^x - 1)) \tilde{J}_X(dx, du).$$

Using the isometry formula,

$$Var^t[A + B] = C + D,$$

where,

$$C := \int_t^{t+\Delta t} \left[(\frac{\partial F}{\partial S} - \theta_u)\sigma S_u \right]^2 du$$

$$D := \int_t^{t+\Delta t} \int_{\mathbb{R}} \left[F(S_{u-}e^x, u) - F(S_{u-}, u) - \theta_u S_{u-}(e^x - 1) \right]^2 \nu(dx) du.$$

The first order condition for the optimal Θ gives the equation:

$$\Theta_t \left[\int_{\mathbb{R}} S_t^2 (e^x - 1)^2 \nu(dx) + \sigma^2 S_t^2 \right] = \frac{\partial F}{\partial S}\sigma^2 S_t^2 + \int_{\mathbb{R}} [F(S_t e^x, t) - F(S, t)] S_t (e^x - 1)\nu(dx),$$

which can be rewritten

$$\Theta_t = \frac{\frac{\partial F}{\partial S}\sigma^2 S_t^2 + \int_{\mathbb{R}} [F(S_t e^x, t) - F(S, t)] S_t (e^x - 1)\nu(dx)}{\int_{\mathbb{R}} S_t^2 (e^x - 1)^2 \nu(dx) + \sigma^2 S_t^2}, \tag{19}$$

which is simply (13). The Θ and Λ ratios therefore coincide, which is an important result.

2.3.2 Δ and Θ ratios

The Δ ratio, which is the derivative of the option price with respect to the underlying price, is often used as a hedging ratio because it measures the sensitivity of the option to the underlying price. Using formula (12),

$$\Delta = \frac{\partial P(S, K, \tau)}{\partial S} = K \frac{1}{2S\pi} e^{-bx'} \int_{\mathbb{R}} e^{iux'} \frac{e^{-\tau(r+\psi_Q(u+ib))}}{iu - b - 1} du. \tag{20}$$

However, with jumps in the price process, this ratio is only optimal in the Black and Scholes setting. In this case, the Δ and Θ ratios coincide. With jumps, the difference is given by the following formula:

$$\Theta - \Delta = K \frac{1}{2S\pi} e^{-bx'} \int_{\mathbb{R}} e^{iux'} \frac{e^{-\tau(r+\psi_Q(u+ib))}}{(iu - b - 1)} \left(\frac{B^{\mathcal{P}(u+ib)}}{iu - b} - 1 \right) du.$$

Figure 3 illustrates this difference, for the four jump processes considered in this article. We let the time to expiry τ vary from 0.5 to 1 for the European put option. We note that the differences between Θ and Δ are almost null for the deeply out-of-the-money options and rise more slowly to zero when the option becomes increasingly in-the-money. When the option is at-the-money, the difference is at its highest. In fact, immediately after the reset dates, the insurer's position is similar to that of an in-the-money or at-the-money put option writer, where the difference between the optimal Θ and Δ ratios is at its highest. As noted by Hardy (2003), the hedging portfolio swings from a long to a short position which makes the hedge very sensitive to price movements, thereby increasing the hedging errors.

2.3.3 Explicit solutions in the Merton Jump-Diffusion model

In this section we show that the optimal Θ ratio can be obtained explicitly in the Merton jump-diffusion model. To do this, we use the Λ expression. In the case of jump diffusion models we have :

$$X_t := at + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{21}$$

where W is a standard Brownian motion, N is a Poisson process with intensity λ, and the (Y_i) are iid random variables having the same distribution as the random variable Y. We assume that all random quantities are independent. Note that in a risk-neutral world, because discounted prices are Q-martingales, we have the EMM

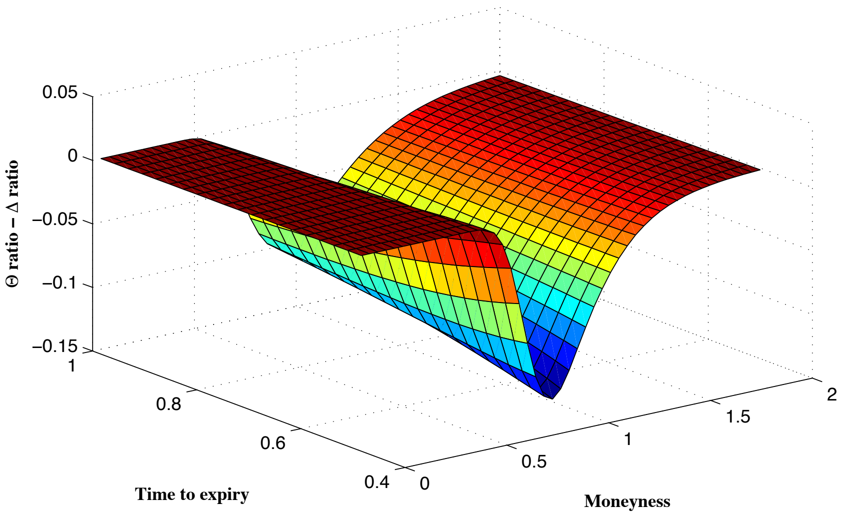
$$a = r - \frac{1}{2}\sigma^2 - \lambda E_Q[e^Y - 1].$$

In the Merton model, the iid random variables Y_i are Gaussian with mean γ and standard deviation δ. In this model the time t price, $F(S_t, t)$, of a call or a put option with strike K, with payoff $[\epsilon(S_T - K)]^+$, is

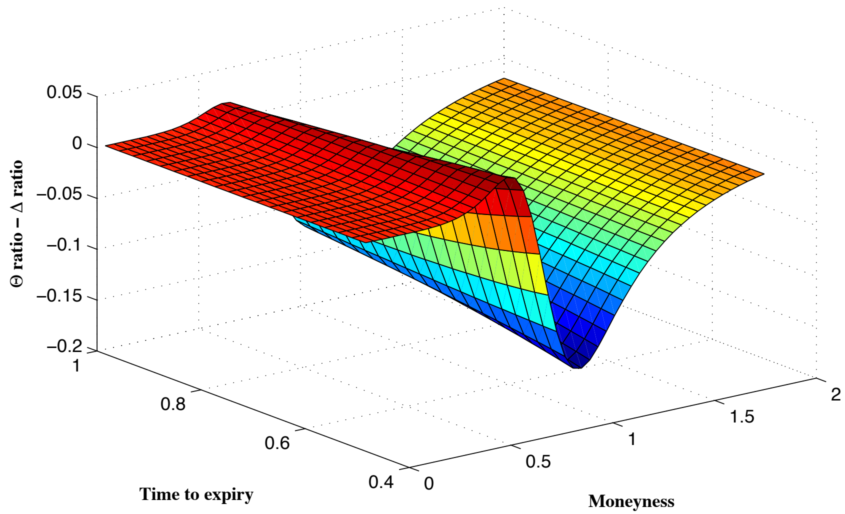
$$F(S, t) = E_Q^t \left[e^{-r(T-t)} [\epsilon(S_T - K)]^+ \right].$$

Figure 3: Difference between the Δ and Θ ratios in some jump models for a put option.

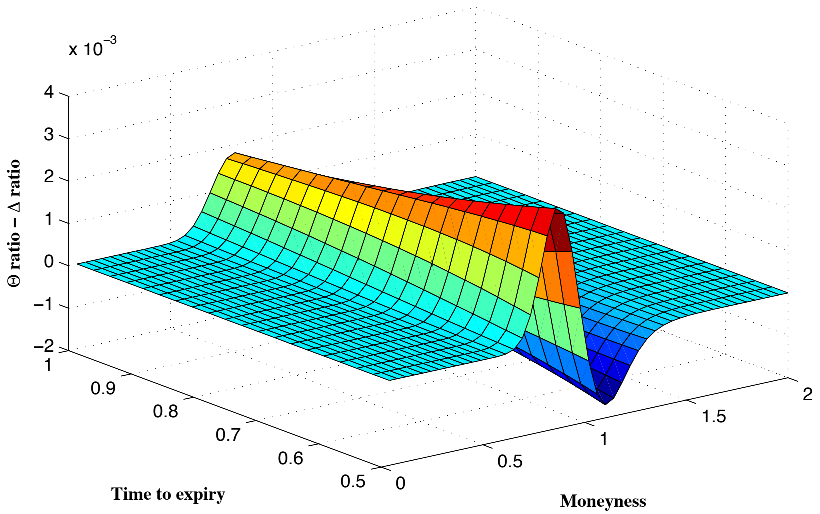
(a) Merton: Parameters used are $\lambda = 1$, $\sigma = 0.16$, $m = -0.2$, and $\delta = 0.05$.



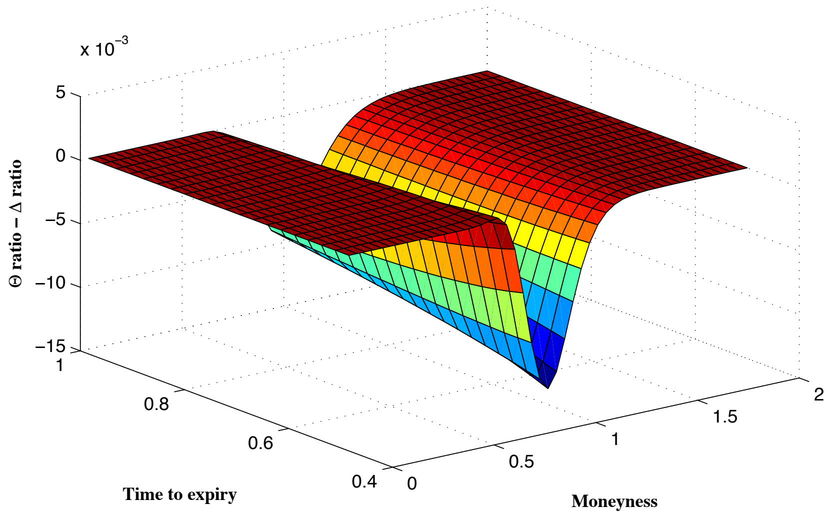
(b) Kou: Parameters used are $\lambda = 1$, $\sigma = 0.16$, $\lambda_1 = 50$, $\lambda_2 = 25$, and $p = 0.3$.



(c) VG: Parameters used are $\theta = -0.008$, $\varsigma = 0.01$, and $\sigma = 0.17$.



(d) CGMY: Parameters used are $C = 2$, $G = 40$, $M = 52$, and $Y = 0.7$.



Conditional on N_T,

$$F(S,t) = E_Q^t\left[E_Q^T\left[\epsilon\left(S_0 e^{at+\sigma W_T+\sum_{i=0}^{N_T} Y_i} - K\right)^+ e^{-r(T-t)}|N_T = n\right]\right],$$

where ϵ is equal to ± 1, 1 for a call, and -1 for a put. There are many ways to present the result, here we choose the following. Define the quantities $\kappa = e^{\gamma+\frac{\delta^2}{2}} - 1$ and $S_n = Se^{n\gamma+\frac{n\delta^2}{2}-\lambda\kappa\tau}$, and let the function $F_M(\tau, S, \sigma, \epsilon)$ denote the Merton (1976) formula for an expiry $\tau = T - t$. Then

$$F_M(\tau, S, \sigma, \epsilon) = \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} F_{BS}(\tau, S_n, \sigma_n, \epsilon),$$

where

$$F_{BS}(\tau, S_n, \sigma_n, \epsilon) = \epsilon S_n \mathcal{N}\Big(\epsilon\frac{\ln\frac{S_n}{K} + (r + \frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\Big) - \epsilon K e^{-r\tau}\mathcal{N}\Big(\epsilon\frac{\ln\frac{S_n}{K} + (r - \frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\Big),$$

is a Black and Scholes (1973) formula for the prices of European call and put options written on an underlying asset whose price at the valuation date t is S_n with volatility σ_n. The function $\mathcal{N}(.)$ is the cumulative probability distribution function for a standardized normal distribution. By introducing the notation $\Sigma = \sqrt{\sigma^2 + \frac{\delta^2}{\tau}}$, we can now state a result that gives the optimal hedging ratio in the Merton jump-diffusion model:

Proposition 3.

$$\begin{aligned}
\Lambda_t = &\left\{\sigma^2 \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\left(\epsilon\frac{S_n}{S}\mathcal{N}\Big(\epsilon\frac{\ln\frac{S_n}{K} + (r + \frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\Big)\right)\right. \\
&+ \frac{1}{S}\Big(\lambda e^{\gamma+\frac{\delta^2}{2}} F_M(\tau, Se^{\gamma+\frac{3\delta^2}{2}}, \Sigma, \epsilon) - \lambda F_M(\tau, Se^{\gamma+\frac{\delta^2}{2}}, \Sigma, \epsilon) \\
&\left.- \lambda(e^{\gamma+\frac{\delta^2}{2}} - 1)F_M(\tau, S, \sigma, \epsilon)\Big)\right\} \times \left\{\sigma^2 + \lambda[e^{2\gamma+2\delta^2} - 2e^{\gamma+\frac{\delta^2}{2}} + 1]\right\}^{-1},
\end{aligned}$$

Proof. See Appendix A. □

In Table 1, we provide the difference between the Θ and Λ ratios in the MJD model. We observe that the Λ ratio leads to the same result as the Θ ratio. Table 2 illustrates the speed of computation of the Θ ratios in the four standard models considered in this article. The main advantage of this formula is that it can be quickly computed on a standard PC, in any Lévy setting with known characteristic exponent.

2.4 Discussion of the Q-measure

The main problem with the risk-neutral measure Q in an incomplete market stems from the fact that it is not unique, which gives rise to the troubling situation in which we can have as many prices as risk-neutral measures. This

Call Option ($\epsilon = 1$)			Put Option ($\epsilon = -1$)		
S	Θ	Absolute error: $\lvert\Theta - \Lambda\rvert$	S	Θ	Absolute error: $\lvert\Theta - \Lambda\rvert$
120	0.91404	1.44328e-15	80	-0.86806	1.99840e-15
100	0.61819	1.22124e-15	90	-0.63912	2.44249e-15
90	0.36088	1.33226e-15	100	-0.38181	5.55111e-16
80	0.13193	5.27355e-16	120	-0.08595	2.35922e-16

Table 1: Difference between Θ in equation (16), obtained via the FFT, and Λ in equation (13) in the MJD model. Parameters used are $K = 98$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 1$, $\gamma = -0.1$, $\delta = 5\%$, and $\tau = 0.5$.

Models	FFT ($N = 4096$)
Merton Kou Variance Gamma CGMY	$\simeq 0.008$ s

Table 2: Fourier performance for optimal Θ ratios (in seconds).

well-known issue has given rise to an abundant literature. A practical solution could be to assume a particular parametric model for financial prices and, given option quotes, find the parameters that best fit the market data. Another solution that we consider in this article is to choose the Esscher measure such that discounted prices (or gain processes) are Q-martingales. It is then possible to link the real-world parameters to the risk-neutral world parameters. For some stochastic processes, the temporal laws of a particular type remain of the same type in both universes.

Another interesting point is the choice of the universe for constructing hedging strategies. It seems more natural to consider the real world than the risk-neutral one. The choice of the Esscher risk-neutral measure seems very well suited to this case because of the link it allows between historical and risk-neutral universes. This solution is used in section 3.

The Esscher measure Q_k with the parameter k is defined by the Radon-Nikodym derivative

$$\frac{dQ_k}{d\mathcal{P}} \mid_{\mathcal{F}_t} = \frac{e^{kX_t}}{E_{\mathcal{P}}[e^{kX_t}]}.$$

The parameter k^* that satisfies the martingale restriction defines the risk-neutral measure. The martingale condition is therefore $S_0 = E_{Q_k}[S_0 e^{-rt} e^{X_t}]$. Thus giving,

$$\begin{aligned} S_0 &= S_0 e^{-rt} \int_\Omega \frac{e^{(k+1)X_t}}{E_{\mathcal{P}}[e^{kX_t}]} d\mathcal{P} \\ &= S_0 \exp\Big\{ - \big(r + \psi_{\mathcal{P}}(k+1) - \psi_{\mathcal{P}}(k)\big)t \Big\}. \end{aligned}$$

The chosen risk-neutral measure Q_{k^*} comes from the parameter k^* such that

$$r + \psi_{\mathcal{P}}(-i(k^*+1)) - \psi_{\mathcal{P}}(-ik^*) = 0,$$

which is the martingale relation. The characteristic triplet of a Lévy measure (B, C, ν) in the historical world with the probability measure $\mathcal{P}$ and under the equivalent Esscher martingale Q_{k^*} is, see Shiryaev (1999)

$$\begin{aligned} B_{Q_{k^*}} = B_{\mathcal{P}} + k^* C_{\mathcal{P}} + \int_{|x|\le 1} & x(e^{k^* x} - 1)\nu_{\mathcal{P}}(dx) \\ C_{Q_{k^*}} &= C_{\mathcal{P}} \\ \nu_{Q_{k^*}}(dx) &= e^{k^* x}\nu_{\mathcal{P}}(dx). \end{aligned}$$

2.4.1 Hedging in Practice

We now have all the elements required to define our hedging strategy. We note, however, that two errors may arise when implementing the strategy in practice. The first error arises because the hedging portfolio is actually monitored in discrete time. The second error arises from incomplete markets. We also have to take mortality into account.

Using actuarial notations, we introduce ${}_{\bar{y}|h}q_y$, for any $y \ge x$ and $\bar{y} \ge 0$, as the probability that the policyholder survives during the next $\bar{y}$ years and dies in the subsequent period of length h. Let us denote by $H(t)$ the hedging portfolio at time t just after rebalancing, and by $H(t^-)$, the portfolio just before rebalancing. Following Hardy (2003) the actual hedging error, HE, is

$$\mathrm{HE}(t) = \mathrm{H}(t) - \mathrm{H}(t^-) + {}_{t-h|h}q_x[G_t - A_t]^+.$$

We analyze the pertinence of a hedging strategy through accumulated discounted loss values (ADL). We denote by $\mathcal{L}$ the loss function defined as the sum of three components: the hedging error, the transaction cost, and the fees paid to the insurer

$$\mathcal{L}_t = \mathrm{HE}(t) + \mathrm{TC}_t - {}_tp_x\mathcal{M}_t.$$

The transaction costs due at the end of the t^{th} period denoted by TC_t are proportional to the absolute variation in the amount needed to adjust the hedging portfolio:

$$\mathrm{TC}_t = \mathcal{C}S_t\left|\Psi_t - \Psi_{t-h}\right|.$$

Where Ψ is the number of the risky asset invested in the hedging portfolio. The amount of fees is

$$\mathcal{M}_t = mA_t,$$

where m is determined in equilibrium as a result of equation (6).

3 Numerical Illustration

In this section, we illustrate our previous developments. In the contracts considered, the policyholder is assumed to be 40 years old at the contract's inception. The constant interest rate is set at $r = 6\%$; the transaction costs have a multiplicative coefficient $\mathcal{C} = 0.2\%$; and the initial guarantee is 80% of the initial account value, which is assumed to be equal to the reference equity portfolio value at the contract's inception with a value of USD 100.

Mortality

We use a Gompertz-Makeham mortality law and the US mortality parameters obtained by Melnikov and Romaniuk (2006): The conditional survival probability is therefore:

$${}_tp_x = \exp\left\{-At - \frac{Bc^x(c^t-1)}{\ln c}\right\}.$$

with

$$A = 9.5666\times10^{-4} \quad B = 5.162\times10^{-5} \quad c = 1.09369.$$

Financial prices processes

Using the moment matching method, we fit the Merton, CGMY, and GBM models to the monthly total return S&P 500 index prices, in US dollars, observed from 01/31/1956 to 05/30/2014. The rounded annual parameter estimates are shown in Table 3, together with the four first central moments.

Models	Merton	CGMY	GBM
Estimates	$\mu = 0.1227$ $\sigma = 0.1329$ $\lambda = 0.1769$ $\gamma = -0.1500$ $\delta = 0.0204$ $p = 0.3$	$\mu = 0.2799$ $\mathtt{C} = 0.6235$ $\mathtt{G} = 21.0775$ $\mathtt{M} = 39.5137$ $\mathtt{Y} = 0.8$	$\mu = 0.0962$ $\sigma = 0.1473$
Mean	0.0962	0.0962	
Std	0.1473	0.1473	
Skewness	−0.1969	−0.1969	0
Excess Kurtosis	0.2110	0.2111	0

Table 3: Annual parameters of the Merton, CGMY, and GBM models obtained by fitting the models to the total return S&P 500 index prices, from 01/31/1956 to 05/30/2014.
Source of the data: Bloomberg®.

We also need to know the processes for the referenced portfolio in the Esscher risk-neutral universe. In accordance with section 2.4, a process that is a MJD in the historical world remains a MJD in the Esscher risk-neutral world. The correspondence between the historical and risk-neutral hatted parameters is easy to obtain, leading to

$$\widehat{\lambda} = \lambda e^{k^*\gamma + \frac{(k^*\delta)^2}{2}}, \quad \widehat{\gamma} = \gamma + k^*\delta^2, \quad \widehat{\delta} = k^*\delta.$$

Pricing

Using equation (6), we compute the fair fee rate according to three different financial price models: the two exponential jump processes MJD and CGMY, and the traditional GBM. We firstly note from Table 4 that the results are very similar for the jump processes, which is not surprising, because the estimates of the four central moments are identical and because the fair fees and the value of the embedded option are, in the end, based on expected values. Secondly, we note that the fair fees and embedded option values are significantly lower in a GBM model. Thirdly, we study the influence of the rollover dates on prices. We vary the first reset date in one year steps and let the second rollover date remain 10 years after the first. We retain the same expiry of 22 years. In Table (4a, 4b, 4c), we observe similar behavior for the three price dynamics: in the MJD and CGMY jump process models, the prices and fees increase for the first eight contracts and then decrease for the last two contracts; while in the GBM case, the prices and fees increase for the first nine contracts, before falling in the last contract. We also note that for the last two contracts, the reset dates are close to the contract expiry dates, which explains why the prices and fees dwindle. We therefore conclude that a reset date effect exists with respect to fees and prices.

3.1 Hedging

To analyze hedging strategy efficiency, we use the ADL and run Monte Carlo simulations with 20,000 sample paths, and a monthly frequency for rebalancing the hedging portfolio. We present the results of the MJD and GBM processes. The other jump processes give similar results and are not reported here. In the risk-neutral world we use the hedging ratio Λ or Θ, and in the historical universe the ratio Θ'. In Table 5, the real-world hedging results are shown on the left-hand side, while the results in the Esscher risk-neutral world are presented on the right-hand side. We introduce this distinction for comparison purpose. An argument put forward by some authors in favor of the risk-neutral world when determining a hedging strategy is that the the $Q-$measure contains more information than the historical measure which is particularly true in empirical studies where estimates of probability laws have to be performed, see Aït-Sahalia and Lo (2000). So we examine in these two situations VaR and Conditional Tail Expectation (CTE) risk measures for the ADL. The results show that there are big differences between the Black and Scholes setting and the jump model, see Table 5. In the historical world, for example, the $\text{VaR}_{99\%} = 7.4394$ in the Merton model, while it is only 2.7754 in the lognormal model. The CTE is 3.8086 in the GBM model, but is 9.5562 in the MJD model. If we now consider the computation in the risk-neutral world, we obtain the same values with the GBM but $\text{VaR}_{99\%} = 7.2243$ and $\text{CTE}_{99\%} = 9.3730$ in the MJD, which are not very different from what is obtained in the real world. So even though the hedging choice in the real-world is more intuitive and satisfactory, the hedging choice in the risk-neutral world, although harder to justify, produces results relatively close to those obtained in the real world. We also note that the economic capital measured by VaR or CTE is slightly lower than that required when hedging in the real world. Table 6 shows the result of a Δ hedging strategy compared to the Θ' strategy. The differences are not very significant, contrary to what was expected. The Δ strategy requires more economic capital than the Θ' strategy. In the presence of jumps, the hedging strategy in the risk-neutral world is less demanding in terms of economic capital than that what is required using a hedging strategy in the real-world. Aside from the fact that our methodology gives operational results, we also obtain a clear appreciation of the choice of models for financial prices. Our methodology provides a very simple and useful rule for insurers and regulators: if the assumption of lognormal financial prices is supported by market data, then a hedging strategy based on this hypothesis is better than that obtained using models with jumps, in the sense that the economic capital required is considerably lower than that required when jumps are present. However, as recalled in the introduction, the lognormal hypothesis is generally rejected using actual market data, so if insurers use this hypothesis for hedging, the model will give them a false feeling of safety because it leads to insufficient amounts of economic capital, which could have severe consequences for the company in the future.

GMAB Maturity: 22 years		
Rollover dates	Merton model	
$(t_1/t_2/t_3)$ (years)	m (bp)	Value
(2/12/22)	22.43	4.5664
(3/13/22)	24.30	4.9379
(4/14/22)	25.77	5.2267
(5/15/22)	26.97	5.4633
(6/16/22)	27.99	5.6652
(7/17/22)	28.87	5.8380
(8/18/22)	29.57	5.9751
(9/19/22)	29.96	6.0505
(10/20/22)	29.69	5.9977
(11/21/22)	27.73	5.6129

(a)

GMAB Maturity: 22 years		
Rollover dates	CGMY model	
$(t_1/t_2/t_3)$ (years)	m (bp)	Value
(2/12/22)	22.43	4.5637
(3/13/22)	24.29	4.9352
(4/14/22)	25.75	5.2239
(5/15/22)	26.95	5.4606
(6/16/22)	27.98	5.6626
(7/17/22)	28.87	5.8357
(8/18/22)	29.56	5.9731
(9/19/22)	29.95	6.0491
(10/20/22)	29.69	5.9971
(11/21/22)	27.73	5.6134

(b)

GMAB Maturity: 22 years		
Rollover dates	GBM model	
$(t_1/t_2/t_3)$ (years)	m (bp)	Value
(2/12/22)	18.64	3.8109
(3/13/22)	20.32	4.1463
(4/14/22)	21.67	4.4153
(5/15/22)	22.82	4.6435
(6/16/22)	23.84	4.8467
(7/17/22)	24.77	5.0305
(8/18/22)	25.58	5.1893
(9/19/22)	26.14	5.3007
(10/20/22)	26.16	5.3045
(11/21/22)	24.69	5.0154

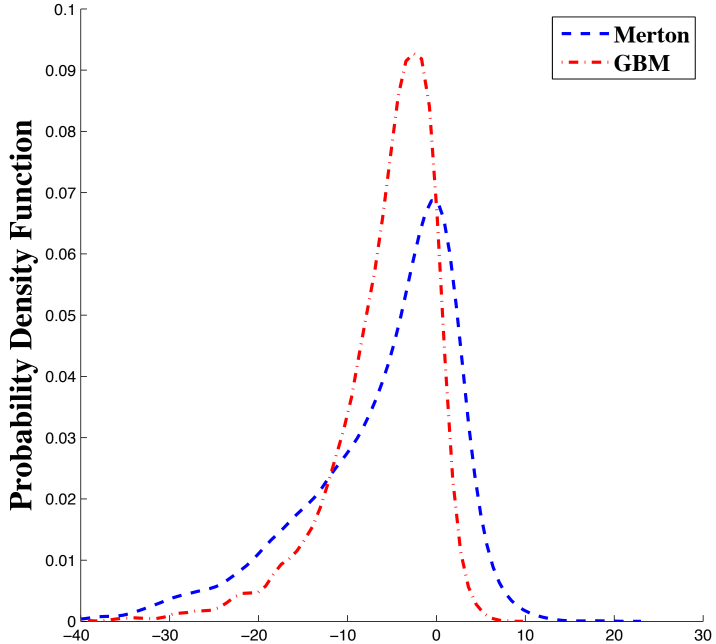
(c)

Table 4: Fair values of the GMAB in the Merton jump-diffusion model (4a), and in the CGMY model (4b), obtained using formula (6), compared with those of the GBM model (4c).

GMAB

Rollover dates: (2/12/22) years

α	Merton model		GBM model	
	VaR_α	CTE_α	VaR_α	CTE_α
50%	−3.5716	0.5272	−4.3349	−1.5308
90%	2.5365	4.5793	0.0340	1.1964
95%	3.8859	6.0512	0.8962	2.0090
97.5%	5.3802	7.5373	1.6091	2.7927
99%	7.4394	9.5562	2.7754	3.8086

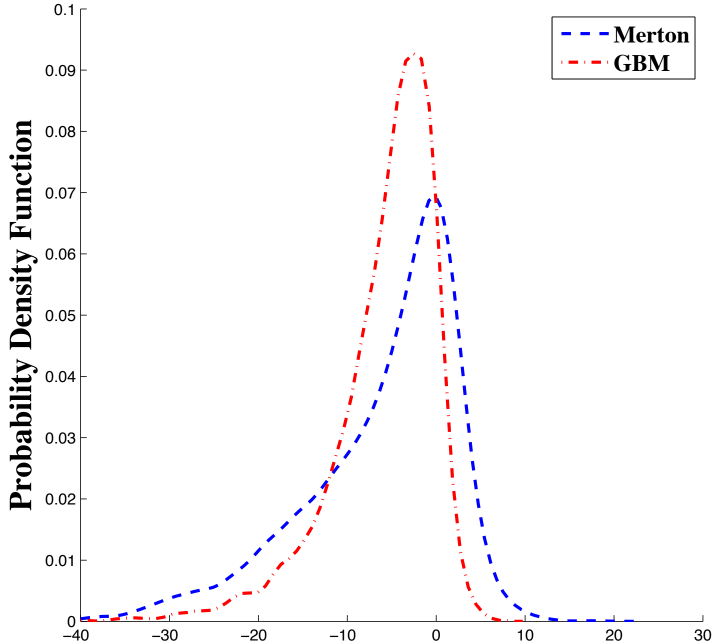


(a) Real world

GMAB

Rollover dates: (2/12/22) years

α	Merton model		GBM model	
	VaR_α	CTE_α	VaR_α	CTE_α
50%	−3.6358	0.4742	−4.3349	−1.530
90%	2.4454	4.4669	0.0340	1.1964
95%	3.7636	5.9005	0.8962	2.0090
97.5%	5.2359	7.3723	1.6091	2.7927
99%	7.2243	9.3730	2.7754	3.8086



(b) Esscher-risk-neutral world

Table 5: Probability distribution functions and risk measures of the ADL for 100 USD premium VAs with GMAB benefits: Hedging using the Θ' ratio (5a) and the Θ ratio (5b), compared with the GBM model.

GMAB				
Rollover dates: (2/12/22) years				
α	Δ Hedging		Θ' Hedging	
	VaR_α	CTE_α	VaR_α	CTE_α
50%	-3.4456	0.7134	-3.5716	0.5272
90%	2.6964	5.0506	2.5365	4.5793
95%	4.2893	6.6697	3.8859	6.0512
97.5%	5.9655	8.2374	5.3802	7.5373
99%	7.9812	10.3335	7.4394	9.5562

Table 6: Risk measures of the ADL, for 100 USD premium VAs with GMAB benefits, obtained via the optimal Θ' ratio (15) and via the Δ ratio (20).

4 Conclusion

In this paper we present a general methodology for pricing, hedging, and managing a GMAB contract with an incorporated ratchet feature. Formula (8) gives the price of this path-dependent contract in a general closed-form expression. We obtain an optimal hedging ratio, which we call the Θ ratio, in an expression well suited for easy computation by FFT formula (16). We show that this ratio is equivalent to the ratio used in Cont and Tankov (2004). We give a quasi-explicit formula for the Θ ratio in the Merton jump-diffusion model. The Θ ratio is easily and quickly obtained in a general Lévy framework. Table 2 shows the related computational time for the Merton, Kou, Variance Gamma, and CGMY processes, which is approximately one millisecond with a standard PC. We also show that the difference between the Δ and Θ ratios highlights the risk of using delta hedging for options, especially when they are at-the-money. The numerical part of our study is not solely presented for illustrative purposes, it also provides results that have important practical implications for the management of GMABs. It shows that changing the rollover dates affects the fees and prices of the embedded option. It also proposes a simple rule for choosing a pertinent hedging strategy. If financial markets support the Gaussian hypothesis for financial asset returns, then the hedging strategy obtained in this setting is the best because it requires less economic capital than the strategy obtained in an environment that assumes jumps in financial prices. Conversely, if jumps are present, which is rare, developing a hedging strategy based on the Black and Scholes model undervalues the economic capital required by the insurer to implement a safe hedging policy.

Appendix

In this appendix, we prove the results stated in the main text.

A The Λ ratio in a Merton Economy

Let us rewrite in the following way the formula giving the Λ ratio in Cont and Tankov (2004)

$$\Lambda_t = \frac{\sigma^2 \frac{\partial F_M}{\partial S}(\tau, S, \sigma, \epsilon) + \frac{1}{S}\int_{-\infty}^{+\infty}[F_M(\tau, Se^x, \sigma, \epsilon) - F_M(\tau, S, \sigma, \epsilon)](e^x - 1)\nu(dx)}{\sigma^2 + \int_{-\infty}^{+\infty}(e^x - 1)^2 \nu(dx)}.$$

Observe that the computation of the above formula is not straightforward although

$$\int_{-\infty}^{+\infty}(e^x - 1)^2 \nu_X(dx) = \lambda[e^{2\gamma + 2\delta^2} - 2e^{\gamma + \frac{\delta}{2}} + 1],$$

from the Laplace transform of Gaussian random variable and that

$$\frac{\partial F_M}{\partial S}(\tau, S, \sigma, \epsilon) = \sum_{n \geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\left\{\epsilon, \frac{S_n}{S}\mathcal{N}\left(\epsilon\frac{\ln\frac{S_n}{K} + (r + \frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\right)\right\}$$

is readily derived. Note that Lemma 1 will be useful for deriving the remaining quantities.

Lemma 1. *Let X be a random variable such that $X \sim \mathcal{N}(\gamma, \delta^2)$, for all $\alpha \in \mathbb{R}$, $\beta > 0$, $\epsilon \in \{-1, 1\}$ and $\eta \geq 0$,*

$$E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X + \alpha}{\beta}\Big)\Big] = e^{\eta\gamma + \frac{(\eta\delta)^2}{2}}\mathcal{N}\Big(\epsilon\frac{\alpha + \gamma + \eta\delta^2}{\sqrt{\beta^2 + \delta^2}}\Big).$$

Proof.

$$\begin{aligned}
E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X + \alpha}{\beta}\Big)\Big] &= \int_{-\infty}^{+\infty} e^{\eta x}\Big(\int_{-\infty}^{+\infty} 1_{y < \epsilon\frac{x+\alpha}{\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\Big)\frac{e^{-\frac{1}{2}\frac{(x-\gamma)^2}{\delta^2}}}{\sqrt{2\pi\delta^2}}dx \\
&= \int_{-\infty}^{+\infty} e^{\eta x}\Big(\int_{-\infty}^{+\infty} 1_{\eta y < \epsilon\frac{\eta x+\eta\alpha}{\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\Big)\frac{e^{-\frac{1}{2}\frac{(x-\gamma)^2}{\delta^2}}}{\sqrt{2\pi\delta^2}}dx \\
&= \int_{-\infty}^{+\infty} e^{x}\Big(\int_{-\infty}^{+\infty} 1_{\eta y < \epsilon\frac{x+\eta\alpha}{\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\Big)\frac{e^{-\frac{1}{2}\frac{(x-\eta\gamma)^2}{(\eta\delta)^2}}}{\sqrt{2\pi(\eta\delta)^2}}dx \\
&= \int_{-\infty}^{+\infty} e^{x}\Big(\int_{-\infty}^{+\infty} 1_{y < \epsilon\frac{x+\eta\alpha}{\eta\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\Big)\frac{e^{-\frac{1}{2}\frac{(x-\eta\gamma)^2}{(\eta\delta)^2}}}{\sqrt{2\pi(\eta\delta)^2}}dx.
\end{aligned}$$

We can write,

$$
\begin{aligned}
&E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X+\alpha}{\beta}\Big)\Big]\\
&\qquad = e^{\eta\gamma+\frac{(\eta\delta)^2}{2}}\int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty}1_{y<\epsilon\frac{x+\eta\alpha}{\eta\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\Big)\frac{e^{-\frac{1}{2}\frac{\left(x-\left(\eta\gamma+(\eta\delta)^2\right)\right)^2}{(\eta\delta)^2}}}{\sqrt{2\pi(\eta\delta)^2}}dx\\
&\qquad = e^{\eta\gamma+\frac{(\eta\delta)^2}{2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}1_{y<\epsilon\frac{x+\eta\alpha}{\eta\beta}}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}\frac{\left(x-\left(\eta\gamma+(\eta\delta)^2\right)\right)^2}{(\eta\delta)^2}}}{\sqrt{2\pi(\eta\delta)^2}}dydx.
\end{aligned}
$$

Hence,

$$
E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X+\alpha}{\beta}\Big)\Big] = e^{\eta\gamma+\frac{(\eta\delta)^2}{2}}E\Big[1_{Y<\epsilon\frac{X+\eta\alpha}{\eta\beta}}\Big].
$$

while $Y\sim\mathcal{N}(0,1)$ is independent from $X\sim\mathcal{N}\left(\eta\gamma+(\eta\delta)^2,(\eta\delta)^2\right)$,we obtain:

$$
E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X+\alpha}{\beta}\Big)\Big] = e^{\eta\gamma+\frac{(\eta\delta)^2}{2}}\mathcal{N}\left(\epsilon\frac{\eta\alpha+\left(\eta\gamma+(\eta\delta)^2\right)}{\sqrt{(\eta\beta)^2+(\eta\delta)^2}}\right).
$$

After simplification,

$$
E\Big[e^{\eta X}\mathcal{N}\Big(\epsilon\frac{X+\alpha}{\beta}\Big)\Big] = e^{\eta\gamma+\frac{(\eta\delta)^2}{2}}\mathcal{N}\left(\epsilon\frac{\alpha+\gamma+\eta\delta^2}{\sqrt{\beta^2+\delta^2}}\right),\qquad \forall\ \epsilon\in\{-1,1\}.
$$

□

Proposition 4. *We have:*

$$
\int_{-\infty}^{+\infty}F_M(\tau,Se^x,\sigma,\epsilon)e^x\nu_X(dx) = \lambda e^{\gamma+\frac{\delta^2}{2}}F_M(\tau,Se^{\gamma+\frac{3\delta^2}{2}},\Sigma,\epsilon),
$$

Proof.

$$
\int_{-\infty}^{+\infty}F_M(\tau,Se^x,\sigma,\epsilon)e^x\nu_X(dx) = \sum_{n\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}E_Q[F_{BS}(\tau,S_ne^X,\sigma_n,\epsilon)e^X].
$$

But

$$
\begin{aligned}
E_Q[F_{BS}(\tau,S_ne^X,\sigma_n,\epsilon)e^X] &= \epsilon S_nE_Q\Big[e^{2X}\mathcal{N}\Big(\epsilon\frac{X+\ln\frac{S_n}{K}+(r+\frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\Big)\Big]\\
&\quad -\epsilon Ke^{-r\tau}E_Q\Big[e^{X}\mathcal{N}\Big(\epsilon\frac{X+\ln\frac{S_n}{K}+(r-\frac{\sigma_n^2}{2})\tau}{\sigma_n\sqrt{\tau}}\Big)\Big].
\end{aligned}
$$

Using Lemma 1,

$$
\begin{aligned}
&\lambda^{-1}E_Q[F_{BS}(\tau,S_ne^X,\sigma_n,\epsilon)e^X]\\
&\qquad = \epsilon S_ne^{2\gamma+2\delta^2}\mathcal{N}\Bigg(\epsilon\frac{(\gamma+2\delta^2)+\ln\frac{S_n}{K}+(r+\frac{\sigma_n^2}{2})\tau}{\sqrt{(\sigma_n^2+\frac{\delta^2}{\tau})\tau}}\Bigg)\\
&\qquad -\epsilon e^{\gamma+\frac{\delta^2}{2}}Ke^{-r\tau}\mathcal{N}\Bigg(\epsilon\frac{\gamma+\delta^2+\ln\frac{S_n}{K}+(r-\frac{\sigma_n^2}{2})\tau}{\sqrt{(\sigma_n^2+\frac{\delta^2}{\tau})\tau}}\Bigg).
\end{aligned}
$$

Adding and subtracting $\frac{\delta^2}{2}$,

$$\lambda^{-1}E_Q[F_{BS}(\tau, S_n e^X, \sigma_n, \epsilon)e^X]$$

$$= \epsilon S_n e^{2\gamma+2\delta^2}\mathcal{N}\left(\epsilon\frac{(\gamma+2\delta^2)+\ln\frac{S_n}{K}-\frac{\delta^2}{2}+(r+\frac{\sigma_n^2}{2}+\frac{\delta^2}{2\tau})\tau}{\sqrt{(\sigma_n^2+\frac{\delta^2}{\tau})\tau}}\right)$$

$$-\epsilon e^{\gamma+\frac{\delta^2}{2}}Ke^{-r\tau}\mathcal{N}\left(\epsilon\frac{\gamma+\delta^2+\ln\frac{S_n}{K}+\frac{\delta^2}{2}+(r-\frac{\sigma_n^2}{2}-\frac{\delta^2}{2\tau})\tau}{\sqrt{(\sigma_n^2+\frac{\delta^2}{\tau})\tau}}\right).$$

with $\Sigma_n = \sqrt{\sigma_n^2+\frac{\delta^2}{\tau}}$,

$$\lambda^{-1}E_Q[F_{BS}(\tau, S_n e^X, \sigma_n, \epsilon)e^X]$$

$$= \epsilon S_n e^{2\gamma+2\delta^2}\mathcal{N}\left(\epsilon\frac{\ln\frac{S_n}{K}+\gamma+\frac{3\delta^2}{2}+\left(r+\frac{\Sigma_n^2}{2}\right)\tau}{\sqrt{\Sigma_n^2\tau}}\right)$$

$$-\epsilon e^{\gamma+\frac{\delta^2}{2}}Ke^{-r\tau}\mathcal{N}\left(\epsilon\frac{\ln\frac{S_n}{K}+\gamma+\frac{3\delta^2}{2}+\left(r-\frac{\Sigma_n^2}{2}\right)\tau}{\sqrt{\Sigma_n^2\tau}}\right).$$

In an equivalent way,

$$\lambda^{-1}E_Q[F_{BS}(\tau, S_n e^X, \sigma_n, \epsilon)e^X]$$

$$= \epsilon S_n e^{2\gamma+2\delta^2}\mathcal{N}\left(\epsilon\frac{\ln\frac{S_n e^{\gamma+\frac{3\delta^2}{2}}}{K}+\left(r+\frac{\Sigma_n^2}{2}\right)\tau}{\sqrt{\Sigma_n^2\tau}}\right)$$

$$-\epsilon e^{\gamma+\frac{\delta^2}{2}}Ke^{-r\tau}\mathcal{N}\left(\epsilon\frac{\ln\frac{S_n e^{\gamma+\frac{3\delta^2}{2}}}{K}+\left(r-\frac{\Sigma_n^2}{2}\right)\tau}{\sqrt{\Sigma_n^2\tau}}\right).$$

Leading to

$$\lambda^{-1}E[F_{BS}(t, S_n e^X, \sigma_n, \epsilon)e^X] = F_{BS}(\tau, S_n e^{\gamma+\frac{3\delta^2}{2}}, \Sigma_n, \epsilon).$$

With $\Sigma = \sqrt{\sigma^2+\frac{\delta^2}{\tau}}$,

$$\int_{-\infty}^{+\infty} F_M(\tau, Se^x, \sigma, \epsilon)e^x\nu_X(dx) = \lambda e^{\gamma+\frac{\delta^2}{2}}F_M(\tau, Se^{\gamma+\frac{3\delta^2}{2}}, \Sigma, \epsilon),$$

where

$$F_M(\tau, S, \Sigma, \epsilon) = \sum_{n\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}F_{BS}(\tau, S_n, \Sigma_n, \epsilon).$$

□

Using Lemma 1 and the similar above rationale, it can be shown that

$$\int_{-\infty}^{+\infty} F_M(\tau, S, \sigma, \epsilon)(e^x-1)\nu_X(dx) = \lambda(e^{\gamma+\frac{\delta^2}{2}}-1)F_M(\tau, S, \sigma, \epsilon),$$

and with

$$\int_{-\infty}^{+\infty} F_M(\tau, Se^x, \sigma, \epsilon)\nu_X(dx) = \lambda F_M(\tau, Se^{\gamma+\frac{\delta^2}{2}}, \Sigma, \epsilon),$$

we obtain the result in Proposition 3.

References

Aït-Sahalia, Y. and Lo, A. W. (2000). Nonparametric risk management and implied risk aversion. *Journal of econometrics*, 94(1):9–51.

Bacinello, A. (2003). Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. *Journal of Risk and Insurance*, 70(3):461–487.

Ballotta, L. (2005). A Lévy process-based framework for the fair valuation of participating life insurance contracts. *Insurance: Mathematics and Economics*, 37(2):173–196.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659.

Boyarchenko, S. and Levendorskiĭ, S. (2000). Option pricing for truncated Lévy processes. *International Journal for Theory and Applications in Finance*, 3(3):549–552.

Carr, P., Geman, H., Madan, D. B., and Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75(2):305–332.

Carr, P. and Madan, D. B. (1998). Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2:61–73.

Cherubini, U., Della Lunga, G., Mulinacci, S., and Rossi, P. (2010). *Fourier transform methods in finance*, volume 524. John Wiley & Sons.

Cont, R. and Tankov, P. (2004). *Financial Modelling with Jump Processes.* Chapman & Hall/CRC Press, London, 2 edition.

Dhaene, J., Kukush, A., Luciano, E., Schoutens, W., and Stassen, B. (2013). On the (in-) dependence between financial and actuarial risks. *Insurance: Mathematics and Economics*, 52(3):522–531.

Gerber, H. U., Shiu, E. S., and Yang, H. (2013). Valuing equity-linked death benefits in jump diffusion models. *Insurance: Mathematics and Economics*, 53(3):615–623.

Hardy, M. (2003). *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance.* John Wiley & Sons, Hoboken, New Jersey, USA.

Kassberger, S., Kiesel, R., and Liebmann, T. (2008). Fair valuation of insurance contracts under Lévy process specifications. *Insurance: Mathematics and Economics*, 42(1):419–433.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science*, 48:1086–1101.

Madan, D. B. and Seneta, E. (1990). The variance gamma model for share market returns. *Journal of Business*, 63(4):511–524.

Melnikov, A. and Romaniuk, Y. (2006). Evaluating the performance of gompertz, makeham and lee-carter mortality models for risk-management with unit-linked contracts. *Insurance: Mathematics and Economics*, 39:310–329.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144.

Milevsky, M. A. and Posner, S. E. (2001). The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds. *The Journal of Risk and Insurance*, 68(1):91–126.

Quittard-Pinon, F. and Randrianarivony, R. (2011). Impacts of jumps and stochastic interest rates on the fair costs of guaranteed minimum death benefit contracts. *The Geneva Risk and Insurance Review*, 36(1):51–73.

Shiryaev, A. N. (1999). *Essential of Stochastic Finance: Facts, Models, Theory*, volume 3. World Scientific Company.

Tanskanen, A. J. and Lukkarinen, J. (2003). Fair valuation of path-dependent participating life insurance contracts. *Insurance: Mathematics and Economics*, 33(3):595 – 609.