Central bank in a contagion model

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Abstract

We propose an innovative way to assess central banks interventions by introducing a central bank in a contagion model proposed by Greenwood, Landier, and Thesmar (2015). In a theoretical study, with two banks and one asset, we show that a total amount spent by the central bank to purchase banks assets to lower systemic risk should be allocated to the bank with the higher leverage ratio. With two banks owning two assets, this amount increases with respect to both bank's leverage ratio and size. Our framework is then applied to European banks during the sovereign debt crisis to assess three unconventional monetary policies. We find that a capital injection policy is the most efficient but costly. The Long Term Refinancing Operation provokes an increase of the amount of assets sold by banks and so of systemic risk. However, the purchase assets policy is beneficial, especially when the central bank buys sovereign debts. The resulting optimal assets purchase, highlights that the amount dedicated to Greek banks should be the most important, due to the high exposure of these banks to Hellenic sovereign debt.

Keywords: Central bank, Financial contagion, Systemic risk, Unconventional policy

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1 Introduction

Since 2007, the magnitude of the financial crisis and its associated recession has led to rapid responses from both governments and central banks in order to restore the confidence on the financial markets, support the banking system and avoid the rationing of credit to households and businesses. Indeed, Pérignon, Thesmar, and Vuillemey (2015) show that banks who encounter problems of financial drying up during the periods of crises, present after, a decrease of profitability, an increase in the number of impaired loans, a leverage ratio higher and finally a reduction in the degree of solvency. Thus, ensuring the stability of the financial system become the responsibility of regulators. This is reflected by the shift in banking regulation from Basel II to Basel III. In fact, the microprudential approach of Basel II requires the calculation of a risk measure, which allows to deduct the minimum own funds required. However this is not sufficient to manage systemic risk, hence the implementation of Basel III which proposes a macroprudential approach. This last, similarly to the microprudential approach, is based on the calculation of a global measure of all negative externalities, potentially systemic, imposed by a bank. The bank is then charged for the corresponding amount, thus correcting its incentives for systemic risk-taking (See Benoit, Hurlin, and Pérignon (2016)).

All this has given birth to a fundamental transformation on the central banks role who have seen their field of competence extended beyond their initial mission. By the way, using a dynamic general equilibrium model, Angelini, Neri, and Panetta (2011) argue that it is efficient when the central bank cooperates with the macro-prudential authority, working for objectives beyond price stability in order to enhance the overall stability of the economy. In fact, some central banks, such as the European Central Bank, now benefit from new roles such as lender of last resort. Freixas, Parigi, and Rochet (2000) clearly show through a ”too-big-to-fail” policy modeling, that when a bank occupying a key position in a banks network becomes insolvent, the central bank must intervene by injecting liquidity in the banking system to prevent the waves of bankruptcies which could have effects on other banks.[1] Indeed, the idea that a central bank should provide liquidity to support the financial system goes back to the 19th century work of Henry Thornton and Walter Bagehot. The latter prescribed in 1873 that the central bank should lend freely against a good collateral.[2]

The central banks should thus have a range of mechanisms to feed the liquidity of financial markets, in periods of crisis, in order to restore financial stability. So, if the traditional instruments [3] do not succeed to restore the situation and to prevent the effects of the contagion, central banks have recourse to non-standard (unconventional) policies, such as the direct intervention on the financial market: they purchase financial instruments to act on the yield curve or to simulate systematically the credit market. However the issue during the financial crises is not only to intervene, but also to choose the appropriate time and way to do so. As well, the central banks should resort to targeted interventions by using tools that are well thought out. Actually, Taylor (2009) argue that, in the middle of subprime crisis, one reason that caused, prolonged, and worsened it, is the support provided by the government for certain financial institutions and their creditors but not for others in an ad hoc fashion without a clear and understandable framework.

This article proposes a new and simple way to study the role of central banks to mitigate systemic risk. Based on this setup, we are also able to assess the accuracy of European Central Bank interventions during the sovereign debt crisis. Our model, similar to the contagion model proposed by Greenwood et al. (2015), takes as given banks' leverage ratio, assets holding, assets liquidity and equity capitals. It considers then a negative return shock experienced by one or many assets. This shock move away banks from their initial leverage. Banks response to this by selling assets to

[1]Freixas, Parigi, and Rochet (2000) model in their own way the policy "too big to fail" in interpreting it as a policy which helps to save the banks which occupy a key position in the interbank network.

[2]Bagehot did suggest that loans should be made at a high interest rate relatively to the pre-crisis period (Freixas, Giannini, Hoggarth, and Soussa (2000)). However, B. S. Bernanke et al. (2008) argue that this is not relevant nowadays since central banks do not encounter the same limitations in their ability to lend.

[3]The conventional (traditional) instrument of monetary policy in most major industrial economies is the very short term nominal interest rate (B. Bernanke, Reinhart, and Sack (2004)).

keep their leverage ratio constant. These sales generate a decline in the general price level which depends on the liquidity of the assets sold and their amount. Hence, banks holding the fire-sold assets suffer, in the next period , from a decrease in their assets holdings value. The authors, introduce also a new systemic risk measure, the *aggregate vulnerability*, which asses the value of losses in the second period due to the contagion episode. We intervene at this level, by modeling a central bank, which acts, from a certain threshold of vulnerability, by adopting an unconventional monetary policy. The objective being to reduce the systemic risk, the central bank, in our case, minimizes the aggregated vulnerability (in absolute terms) of the system given its budget constraint.

We, first, exploit this framework theoretically : we study an asset purchase policy and we show that, for a financial system composed of two banks holding the same two assets in different proportion, the amount allocated to a bank is larger when both its leverage ratio and its total assets value are high.

In the empirical part of the article, we assess the efficiency of three unconventional monetary policies, namely a Long Term Financing Operation (LTRO), an assets purchase and a capital injection. We apply our framework to European banks during the last Eurozone crisis. We mainly use 2011 stress tests inputs published by the European Banking Authority. Our simulations lead to several key findings: we show that a capital injection policy is the most effective to reduce systemic risk, even if this policy is costly. We also find that an LTRO is not beneficial for banks in such crises because it inflates the size of banks balance sheet, enhancing thus the volume of sold assets following a shock, which finally amplifies the fire sale impact. For the assets purchase policy, our results suggest that this policy is more efficient when the central bank buys the sovereign debts. The resulting optimal assets purchase, highlights that the amount dedicated to Greek banks should be the most important, due to the high exposure of these banks to Hellenic sovereign debt.

Related literature. Our paper is related to several strands of the literature. An increasingly growing literature highlights the contagion in financial market due to fire sale or counterparty risk[4] or both. The pioneers who were interested by financial contagion was Allen and Gale (2000) followed by Freixas, Parigi, and Rochet (2000). They worked on financial networks to show that a complete banks network is more resilient to shocks because the proportion of losses of a specific bank is apportioned between other banks via the interbank contracts.[5] Most important results related to this literature (Gai and Kapadia (2010),Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015)) announced that networks where financial institutions are connected (even indirectly) better resist to shocks because they share the risk. However, beyond a certain level, an extreme shock spreads rapidly which can lead to the collapse of the hole financial system. In addition, The recent literature proposes more and more sophisticated models. Choi (2014) and Caballero and Simsek (2013) propose a model based on a strategic approach in the financial institutions behavior. However this models are completely theoretical and can't be used in an empirical framework which is not the case of Greenwood et al. (2015). Indeed, they propose a simple and easily calibratable model using available data and which fit with our objective. By the way, Duarte and Eisenbach (2015) used this framework by adapting it to a panel setting to estimate vulnerability in U.S market and Capponi and Larsson (2015) explicitly extend it in a network model. They also add a non financial sector and market clearing.

Our paper is also directly related to the literature on central banks interventions, specially after the last crisis. In fact, Many papers argue how effective was the governments and central banks actions. Lenza, Pill, and Reichlin (2010) and Hesse and Frank (2009) show that non-standard measures have played a quantitatively significant role in stabilising the financial sector and economy after the collapse of Lehman Brothers. This is also in line with the results of Allen et al. (2009) and Freixas, Martin, and Skeie (2011) who argue that the central bank action increases the effectiveness of the interbank markets and reduces the risk of liquidity. However,

[4] In its survey, Upper (2011) confirms that the interbank loans represent an important fraction of banks balance sheet in many European countries. It was around 29% of total assets of Swiss banks in 2005 and around 25% of total assets of German banks.

[5] See Allen, Carletti, and Gale (2009) for a survey.

Taylor(2009), provide empirical evidence that government actions and interventions prolonged and worsened the financial 2008 crisis, inter alia, due to an unsuitable distribution of aids between financial institutions. Angeloni and Wolff (2012) show also that the LTRO following the sovereign debt had no material effect on banks' stock market value.

Conceptually our paper is most closely to Georg and Poschmann (2010). In fact, they introduce a central bank in a contagion model [6] and show that its activity enhances financial stability.

The rest of the paper is organized as follows. Section 2 describes Greenwood et al. (2015) model. In section 3, We introduce a central bank in this model and study theoretically an asset purchase policy in case of two banks and two assets. Section 4 is dedicated to the assessment of three different monetary policies using stress tests data of 2011.

[6] In their network model of interbank markets, banks optimize a portfolio of risky investments and riskless excess reserves according to their risk and liquidity preferences

2 Model

2.1 Setup

As in Greenwood et al. (2015), we consider N banks and K assets. Each bank i has a total assets a_{it} and m_{ik} represent the weight of the asset k in the portfolio of the bank i. This bank i is financed with a mix of debt d_{it} and equity e_{it}. Its leverage ratio b_i is then equal to d_{it}/e_{it}.

Now, if we consider the whole banking system, we will have in each date a $N \times N$ diagonal matrix of assets, A_t, such that each diagonal element, a_{it}, represents the total assets of a bank i. B is a $N \times N$ diagonal matrix of Leverage, such that each diagonal element represent the leverage ratio of a bank i, b_i. Finally M , a $N \times K$ matrix, represents the portfolio weights matrix composed of different weights m_{ik}.

2.2 Framework mechanism

An initial exogenous shock, $F_t = (f_{1t} f_{2t}, .., f_{Kt})$, on one or several assets return is transmitted to the banks return R_t following this equation:

$$R_t = MF_t \tag{1}$$

This shock R_t in the banks return move away some banks (those affected by the shock) from their initial leverage ratio. We assume at this stage that banks want to keep their leverage constant and sell for that (or buy if the shock is positive) some of their assets in the next period[7]. The amount of the sold assets is equal to $A_t B R_t$[8].

In fact, let's consider a bank i which has in the first period a total asset a_1, an equity e_1 and a debt d_1 and experience a negative shock r_1. Its balance sheet in periods 1 and 2 is represented in table 1 .

[7]In period of crisis, it's difficult to raise capitals. So, to return to their target leverage, it's easier for banks to sell some assets

[8]In case of large shocks, some elements of the vector $A_t B R_t$ are negative, and some banks can not return to their target leverage. For that, in the empirical implementation, we consider the $max(A_t B R_t, -A_t(1 + R_t))$

t=1

Assets	Liabilities
a_1	e_1
	d_1

end of the first period

Assets	Liabilities
$a'_1 = a_1 + a_1 \times r_1$	$e'_1 = e_1 + a_1 \times r_1$
	$d'_1 = d_1$

t=2

Assets	Liabilities
$a_2 = a'_1 + sales$	$e_2 = e'_1$
	$d_2 = d_1 + sales$

Table 1: bank i balance sheet

As explained above, the bank i will sell a part of its assets to maintain a fixed leverage ratio as follows:

$$\begin{aligned}\frac{d_1}{e_1} &= \frac{d_2}{e_2} \\ &= \frac{d_1 + sales}{e_t + a_t \times r_t}\end{aligned}$$

The value of the sold assets for this bank i is then equal to $\frac{d_1}{e_1} \times a_1 \times r_1$.

In matrix terms, namely if we consider the whole financial system, the total value of the sold assets, following a shock R_t, is $A_t B R_t$.

Now we still need to describe the mechanism of sales followed in this framework. At this level, we assume that banks will sell their assets such that the weight of each asset in their portfolio remain unchanged between *t* and *t+1,* i.e, the weight matrix M still constant over time. The vector of net asset purchases Φ in *t+1* is then expressed as follows:

$$\Phi = M' A_t B R_t \tag{2}$$

Φ is a $K \times 1$ vector that each element represent the amount steam of the sale of each asset by all banks. These assets sales has a price impact in the next period which depends on different assets liquidity. The return of assets in $t+1$ is then:

$$F_{t+1} = L\Phi \tag{3}$$

L is a $K \times K$ matrix of price impact expressed in terms of Amihud ratio for each assets class. For simplicity, this matrix is diagonal, so that the sale of asset k has no effect on asset k'.

Finally, price impact cause spillovers to all banks holding the assets affected by the fire sale in $t+1$. By combining the three equation above, we obtain the effect of all banks assets return in t on returns on *t+1*:

$$R_{t+1} = MF_{t+1} = ML\Phi = MLM'BA_tR_t$$

2.3 Aggregate Vulnerability

An initial negative shock to assets returns $F_t = (f_{1t} f_{2t}, .., f_{Kt})$[9] has repercussion on all financial system via a direct and an indirect effect.

- The direct effect : the shock F_t leads to direct losses on banks assets which will be reduced in total by $1'A_tMF_t$.[10] These losses occur in the first period t when the shock happens.

- The indirect effect : this effect appears only in the period following the shock. In fact, the fire sale, by reducing the value of assets returns, induce new losses in the whole financial system given by $1'A_tMF_{t+1}$. Greenwood et al. (2015) introduce thus a new systemic risk measure, the *aggregate vulnerability*, expressed by:

$$AV = \frac{1'A_tMLM'BA_tMF_t}{E_t}$$

where $E_t = \sum_i e_{it}$, the sum of banks equity in the first period. This aggregate vulnerability represents the fraction of system equity capital lost due to spillovers effect if there was a shock F_t to asset returns.

[9]In the reminder of the paper, we restrict to the case where $R_t < 0$

[10]1 is a $N \times 1$vector of ones

3 Introduction of the central bank

We propose, in this section, to introduce a central bank in the contagion model already described and to study an assets purchase policy. The new purpose of the central bank is to ensure the financial system stability and so, in our context, to minimize the systemic risk under some constraints. To this end, the central bank control the vulnerability of the financial system and from a certain vulnerability level it intervenes, in t, by buying assets from different banks. We assume that the amounts collected by banks will be used to pay a part of their debt. Nevertheless, the amount devoted to assets purchase isn't unlimited. In fact, the central bank can not exceed a certain amount P.

This policy of assets purchase reduces fire sale impact by reducing the quantity of assets that banks should sell in the next period to maintain it's leverage ratio constant. As a consequence, the vulnerability level diminish in $t+1$ and thus the systemic risk. The central bank problem, at this stage, is to allocate the amount P, as appropriately as possible, between banks to minimize *the aggregate vulnerability.*

To see the intuition, let's consider a financial system composed of a bank i. We assume that the vulnerability threshold is reached and the central bank should then intervene by buying assets from this bank, in the end of period t, assuming always that this is done in a way which keeps our matrix weight constant over time. Let's c be the value of the assets purchased. The balance sheet of bank i, in t and *t+1*, explaining this operation, is represented in table 2.

In t

Assets	Liabilities	*central bank*	Assets	Liabilities
a_t	e_t	$\Longrightarrow$	$a'_t = a_t - c$	$e'_t = e_t$
	d_t	*intervention*		$d'_t = d_t - c$

In t+1

Assets	Liabilities
$a_{t+1} = a'_t + a'_t r_t + sales$	$e_{t+1} = e'_t + a'_t r_t$
	$d_{t+1} = d'_t + sales$

Table 2: Bank i balance sheet after an asset purchase policy

In *t+1*, to target Leverage, the bank i will sell a quantity of assets of a value equal to $ba_t r_t + (1 - b_t r_t)c$.
In fact,

$$\begin{aligned} \frac{d_t}{e_t} &= \frac{d_{t+1}}{e_{t+1}} \\ &= \frac{d'_t + sales}{e'_t + a'_t r_t} \end{aligned}$$

and so

$$\begin{aligned} sales &= \frac{d_t}{e_t} \times (e_t + (a_t - c) \times r_t) - d_t + c \\ &= ba_t r_t + (1 - b_t r_t)c \end{aligned}$$

To see the intuition, let's consider that this bank i has a leverage ratio equal to 2 and a total assets value of 10 units and that the central bank should intervenes in t by buying 1 unit of assets. If, at the end of the period, the value of the assets decline of 50% (the shock), in the absence of the central bank intervention, the bank should sells 10 units of its assets. However, following the central bank intervention, it will only sells 8 units.

Now, if we consider the whole financial system composed of N banks and following the same methodology described in the second section, the banks return vector is such that:

$$R_{t+1} = MLM' \times [BA_t R_t + (I_N - diag(B_t R_t)) \times C]$$

where :

- I_N is the identity matrix
- $diag(B_tR_t)$ is a diagonal matrix composed of elements of vector B_tR_t
- $C = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix}$ is the vector of amounts allocated to each bank for assets purchases

The *aggregate vulnerability* is then equal to:

$$AV = \frac{1'(A_t - diag(C))MLM' \times [BA_tR_t + (I_N - diag(B_tR_t))C]}{E_t}$$

By rearranging the terms of this equation, we obtain:

$$AV \times E_t = \sum_n \gamma_n [a_n b_n r_n + (1 - b_n r_n)c_n]$$

with $\gamma_n = \sum_k (\sum_m (a_m - c_m) m_{nk}) l_k m_{nk}$ measures the "connectedness" of bank n as in Greenwood et al. (2015).

Once the aggregate vulnerability determined, we will now determine the amount that should be allocated to each bank to minimize efficiently the systemic risk. Remember that if the shock is negative, the value of the *aggregate vulnerability* is also negative. Our problem is then the following:

$$\begin{aligned} &\underset{C}{\text{maximize}} && AV \\ &\text{under constraints} && \sum_n 1'C = P \quad and \quad C > 0 \end{aligned}$$

First of all, for simplicity reasons, we consider two banks which hold the same asset and then we resolve the problem. Here two cases are possible (See appendix 1):

- **Case 1:** $c_1 = P$ and $c_2 = 0$ if $b_1 > b_2$
- **Case 2:** $c_1 = 0$ and $c_2 = P$ if $b_1 < b_2$

Proposition 1: If two banks hold only one asset (the same), the total amount P will be allocated to the bank which has the highest leverage ratio.

Now if we consider two banks with 2 assets and we resolve our initial problem (see appendix 2), three cases are possible:

- if $(t_{12} - 2t_{11})\pi_1 + ((t_{12} - 2t_{11})P + a_2 t_{12})\Sigma_1 + a_1 t_{11} < (t_{12} - 2t_{22})\pi_2 + (a_1 t_{12} - t_{12}P)\Sigma_2 + a_2 t_{22}$, then $c_1 = P$ and $c_2 = 0$

- if $(t_{12} - 2t_{22})\pi_2 + ((t_{12} - 2t_{22})P + a_1 t_{12})\Sigma_2 + a_2 t_{22} < (t_{12} - 2t_{11})\pi_1 + (a_2 t_{12} - t_{12}P)\Sigma_1 + a_1 t_{11}$, then $c_1 = 0$ and $c_2 = P$

- if $(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2 \neq 0$,

$$c_1 = \frac{(t_{12} - t_{11})\pi_1 + (t_{22} - t_{12})\pi_2 + ((2t_{22} - t_{12})\Sigma_2 - t_{12}\Sigma_1)P + (a_1 t_{11} + a_2 t_{12})\Sigma_1 - (a_1 t_{12} + a_2 t_{22})\Sigma_2}{2 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]}$$

$$c_2 = \frac{(t_{11} - t_{12})\pi_1 + (t_{12} - t_{22})\pi_2 + ((2t_{11} - t_{12})\Sigma_1 - t_{12}\Sigma_2)P + (a_1 t_{12} + a_2 t_{22})\Sigma_2 - (a_1 t_{11} + a_2 t_{12})\Sigma_1}{2 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]}$$

  but here we should also verify that c_1 and c_2 are positive.
  At this stage, if we verify the concavity of the function AV (see Appendix 1), we find that it depends on the leverage ratio and in the composition of the two banks' portfolio.

We consider that:

$t_{ij} = \frac{\partial \gamma_i}{\partial c_j} = -\sum_{k=1}^{2} m_{ik} l_k m_{jk}$

$\pi_i = a_i b_i r_i$

$\Sigma_i = 1 - b_i r_i$

Proposition 2: If one of the two banks is affected by a large shock and, at the same time, it has an important Leverage ratio and a big size, the central bank should allocate all the amount P to this bank (case 1 or 2).

Proposition 3: If the two banks hold the two assets in the same proportions, i.e $m_{11} = m_{21}$, then $(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2 = 0$. In this situation, the total amount P will be allocated to one of the two banks depending on the value of the assets and on the leverage ratio (case 1 or 2).

In fact, the result found with two banks and one asset is a particular case of this more general result.

Proposition 4: In case 3, the amount c_i allocated to the bank i is all the more important that the total assets value a_i of this bank is significant.

In fact, for the first bank ,the partial derivative of c_1 with respect to a_1 is positive:

$$\frac{\partial c_1}{\partial a_1} = \frac{(t_{12} - t_{11}) * b_1 r_1 + t_{11}\Sigma_1 - t_{12}\Sigma_2}{2 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]} > 0$$

[11]

Proposition 5: Always in case 3, the amount c_i allocated to the bank i is all the more important that the leverage ratio b_i of this bank is significant.

In fact, for the first bank, the partial derivative of c_1 with respect to b_1 is also positive:

$$\frac{\partial c_1}{\partial b_1} = \frac{I + II}{4 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]^2} > 0$$

with:

$$I = 2[(t_{12} - t_{11})a_1 r_1 + (t_{12}P - t_{11}a_1 - t_{12}a_2)r_1] \times [(t_{11} - t_{12})\Sigma_1 + (t_{22} - t_{12})\Sigma_2] \quad (4)$$

$$\begin{aligned} II = [2(t_{12} - t_{11})r_1] \times [(t_{12} - t_{11})\pi_1 + (t_{22} - t_{12})\pi_2 + ((2t_{22} - t_{11})\Sigma_2 - t_{12}\Sigma_1)P \\ + (a_1 t_{11} + a_2 t_{12})\Sigma_1 - (a_1 t_{12} + a_2 t_{22})\Sigma_2] \quad (5) \end{aligned}$$

[11] the same exercise can be done with the second bank.

4 Policies simulation

As an extension of 2008 financial crisis, the sovereign crisis of euro zone has unleashed in late 2009, by the new Greek government announcement reporting a huge deficit. This deficit was twice as much as announced previously (Jeanneret, Chouaib, et al. (2015)). The crisis was then extended to two others European country: Ireland then Portugal. These shocks have also threaten larger economies that could have jeopardized the Euro zone survival, namely Spain and Italy. This situation created a collective panic among investors, for whom a default of a Eurozone country was not conceivable due to the common market and the unique currency, which presupposes a substantial support from European partners. This uncertainty in financial market required an unprecedented intervention from the European Central Bank which first tried to restore financial stability by using its traditional measures. However, this was not sufficient, which pushed the ECB to have recourse to a battery of non-conventional policies. That's why, we propose in this section an empirical implementation of our framework, based on 2011 Stress tests data. In fact, our theoretical framework is only limited to two banks which hold one or two assets because with more banks and more assets, it will be more complicated. We choose then to simulate three unconventional policies and to compare them. We first start by Long-Term Refinancing Operations (LTRO), we then simulate an asset purchase policy and finally we study a capital injection policy.

4.1 Data description

We gather 3 types of data: the EU 2011 stress test inputs, shocks on PIIGS (Portugal, Ireland, Italy, Greece and Spain) sovereign debt and assets liquidity.

4.1.1 Stress Test data

Since 2011, the European Banking Authority (EBA) has conducted three stress tests. The first ones was in 2011, the second ones in 2014 and the last ones in 2016. We propose, in this framework, to use those of 2011 because they represent the real situation of banks during the crisis. Therefore, we can evaluate the effectiveness of various policies used by the ECB at this moment. In fact, the EBA published in July 2011 EU wide stress test of 90 banks in 21 countries, representing 65% of total assets of the European banking sector.[12] This stress tests exposed detailed balance sheets of stressed banks. Table 3 gives the summary statistics for our sample of balance sheet data.

The assets Matrix A is directly derived from EBA data by considering the sum of all exposures. These exposures are divided in different blocks reflecting the main risks in banks' balance sheets. This allow us to obtain the weight matrix M. We consider thus that M is composed of 10 asset classes: retail loans, corporate loans, commercial real estate, PIIGS sovereign debt (5 classes), other European country sovereign debts and a final class regrouping the remained exposures.

For the matrix B, we don't have values of leverage ratios in stress tests, we then calculate them using equities and assets values, such as, for each bank i, b_i is equal to $\frac{a_i - e_i}{e_i}$. However we impose a Leverage cup of 50 in our sample to not have results greatly influenced by extreme values. Furthermore, targeting a very high leverage is not realistic. This threshold is applied for 7 banks.[13]

4.1.2 Shocks calibration

Data used to calibrate shocks, which represent a reduction in PIIGS sovereign debt value, are from Bloomberg. In fact, we extract PIIGS yield of Government Bond 10Y[14] from 31 December 2010 to 31 December 2011 and we then calculate, for each bond, the variation between its value in 31 December 2010 and the maximum value reached in 2011. Thus we apply shocks of -65% in Greece sovereign debt, -53% in Portugal sovereign debt, -34% in Ireland sovereign debt, -33% in Italy sovereign debt and -19% in Spain sovereign debt.

[12]See http://www.eba.europa.eu/risk-analysis-and-data/eu-wide-stress-testing/2011/results.

[13]We consider that imposing a leverage cup of 30 as in Greenwood et al. (2015) is too strict in such a period. We choose then 50 to only remove outlier values.

[14]Bonds issued by national governments in foreign currencies are normally referred to as sovereign bonds. The yield required by investors to loan funds to governments reflects inflation expectations and the likelihood that the debt will be repaid (Bloomberg).

4.1.3 Assets liquidity

The last input we need is the value of assets liquidity (the matrix L). As in Greenwood et al. (2015), we consider the same liquidity value for all assets classes, such as $L = 10^{-3}I$.[15] It corresponds to 10 basis points price change per 10 billion euros of trading imbalances.

4.2 Long-Term Refinancing Operations

By the LTRO, the European Central Bank provided liquidity to Euro zone banks which were suffering from a lack of access to the interbank market. The aim of these loans was to avoid credit crunch and to support ongoing growth. The LTRO, accorded by the ECB, on December 2011 and February 2012, accounted for more than 1000 €billion (Enrich and Forelle (2012)).

4.2.1 Methodology

In this section, we propose to study a LTRO of €1000 billion by allocating this amount, as appropriately as possible, between banks to maximize the *aggregate vulnerability*. As before, to explain the framework, we first consider a financial system composed of a bank i and we assume that in t the central bank should intervene by granting LTRO. Let's c be the value of the LTRO attributed to the bank i. However here, we suggest to modify the composition of the initial matrix weight considered before. In fact, the aim of the LTRO was to avoid credit crunch, we suggest thus that this amount collected by the bank will be used to buy 3 kind of assets classes, namely retail loans, corporate loans and commercial real estate, proportionally to its initial holding of this assets classes. To see the intuition, consider the bank i with holding 30% of retail loans, 20% of corporate loans, 10% of commercial real estate and 45% of the other assets classes. If the bank receives an amount c of LTRO from the central bank, it will buys $\frac{30\%}{30\%+20\%+10\%} \times c$ of retail loans, $\frac{20\%}{30\%+20\%+10\%} \times c$ of corporate loans

[15] I is a 10×10 identity matrix

and $\frac{10\%}{30\%+20\%+10\%} \times c$ of commercial real estate. The balance sheet of bank i, in t and *t+1*, explaining this kind of central bank refinancing, is represented in Table 4.

In t

$$\begin{array}{l|r} \text{Assets} & \text{Liabilities} \\ \hline a_t & e_t \\ & d_t \end{array} \quad \begin{array}{c} \textit{central bank} \\ \Longrightarrow \\ \textit{intervention} \end{array} \quad \begin{array}{l|r} \text{Assets} & \text{Liabilities} \\ \hline a'_t = a_t + c & e'_t = e_t \\ & d'_t = d_t + c \end{array}$$

In t+1

Assets	Liabilities
$a_{t+1} = a'_t + a'_t r_t + sales$	$e_{t+1} = e'_t + a'_t r_t$
	$d_{t+1} = d'_t + sales$

Table 3: Bank i balance sheet after the LTRO

In t+1, always with the aim of targeting leverage ratio, the bank i will sell a quantity of assets of a value equal to $ba_t r_t + (b_t r_t - 1)c$.

Now, if we consider the whole financial system composed of N banks and following the same methodology described in the second section the banks return vector is such that[16]:

$$R_{t+1} = MLM' \times [BA_t R_t + (diag(B_t R_t) - I_N) \times C]$$

The *aggregate vulnerability* is then equal to:

$$AV = \frac{1'(A_t + diag(C))MLM' \times [BA_t R_t + (diag(B_t R_t) - I_N)C]}{E_t}$$

4.2.2 Results and analysis

A LTRO implementation using our framework[17], shows that this policy is not efficient in such a crisis. In fact, according to our model, LTRO inflates the size of banks balance sheet, enhancing thus the volume of sold assets arsing from a shock, which amplify the fire sale impact. This results, in our case, in an increase in the value of

[16]The weight matrix M used here is calculated by considering the new amounts allocated to the purchase of the three first assets classes, as described above.

[17]We add in the empirical part, a new constraint, such that the amount allocated to bank i should be lower than it's total asset ($c_i < a_i$), because we won't to nationalize banks.

the *Aggregate vulnerability* by 22%. By the way, Angeloni and Wolff(2012) confirmed that the December LTRO had no beneficial effect on banks' stock market values.
The largest banks that should benefit from the LTRO are in most cases Greek, Italian, Portuguese and Spanish ones. These banks are highly affected by the shock and at the same time, in most cases, had a high Leverage ratio.

4.3 Assets purchase policy

During the last crisis, many central banks noticed that their governments were no longer able to revitalise the economy properly because they were paralyzed by a significant debt burden. These central banks decided then to increase the size of their balance sheets by a financial assets purchase, namely the ECB, which launched many assets purchase programs as the Outright Monetary Transactions, the Securities Market Program or whether the Quantitative Easing program in 2015 to fulfill price stability mandate. The last one combined monthly asset purchases to amount to €60 billion, which was carried out until at least September 2016 (ECB (2015)). The total amount of these assets purchase program exceeded thus €1000 billion.

So we propose in this section too, to allocate the €1000 billion optimally among the banking system. We remind that the *aggregate vulnerability* expression, in this case, was determined in the first section by assuming that assets are purchased in the same proportions presented in the matrix M :

$$AV = \frac{1'(A_t - diag(C))MLM' \times [BA_tR_t + (I_N - diag(B_tR_t))C]}{E_t}$$

The assets purchase policy has beneficial effect on banks vulnerability. In fact, using our framework, we find that such a policy reduces the aggregate vulnerability by 40%. Table 5 reports the optimal asset purchase policy for each bank. We only represent the 10 largest banks, ranked by the size of the amount allocated by the central bank to buy assets from each. Here also the banks taking the leading positions in this ranking are in most Greek, Italian, Portuguese and Spanish.

Furthermore, from figure 1, we can note that the amount that should be dedicated to the Greece is the most important, which may be expected given the high shock that affected the Greek banks. Furthermore, these banks have a high exposure to Hellenic sovereign debt. Italy, Portugal and Spain are followed by Cyprus. In fact, although the weight of the Cypriot economy is negligible in the euro area (0.2% of total GDP), the size of the financial sector is considerable in relation to the country's economy and is equivalent to eight times its GDP. Moreover, the two largest Cypriot banks (ranked in top 15 of our banks classification) had major operations in Greece, so the Greek part of the operation and the bond holdings caused them a lot of damage (Economist (2015)). Germany and Belgium are also among the countries that should benefit from the asset purchase program. Indeed, Germany has a significant banking system with some banks which are highly levered and at same time affected by shocks to sovereign debts. However, for Belgium, Dexia, which is a Belgian bank with an important size and a high Leverage ratio, has first been weakened by the subprime crisis and the euro crisis then makes its situation worse due to its high exposure to PIIGS debt.

There are several potential approaches to unconventional monetary policy depending, among other things, on which kind of assets are purchased. That's why we propose now, to take as input the optimal amounts granted to Eurozone banks, as calculated above, and determine which assets the central bank should buy to further minimize the *aggregate vulnerability*. The result suggests that central bank should buy more PIIGS debt (Table 6) which will reduce further the *aggregate vulnerability* by 53%. In fact, purchasing sovereign debt, if it is possible, will reduce in the next period the amount of the sold assets and so the fire sale impact which, in turn, reduce the systemic risk.

4.4 Equity injection policy

We propose in this section to study an equity injection policy, departing from the same amount P (1000 billion of euros). This sum of money will be used to inject, optimally, capitals to banks to maximize as much as possible the *aggregate vulnerability*. Here, we suppose that banks use the capitals injected to repay a part of their debt.

4.4.1 Methodology

To determine the expression of the Aggregate vulnerability in this case, we follow the same methodology described above: we first consider a financial system composed of one bank i and assume that in t the central bank should intervenes by injecting a capital c. The balance sheet of bank i, in t and $t+1$, is exposed in table 7.

In t

Assets	Liabilities		Assets	Liabilities
a_t	e_t	*central bank* $\Longrightarrow$	$a'_t = a_t$	$e'_t = e_t + c$
	d_t	*intervention*		$d'_t = d_t - c$

In t+1

Assets	Liabilities
$a_{t+1} = a_t + a_t r_t + sales$	$e_{t+1} = e'_t + a'_t r_t$
	$d_{t+1} = d'_t + sales$

Table 4: Bank i balance sheet after a capital injection

In t+1, to return to its target leverage, the bank i sells assets for an amount of $ba_t r_t + (1 + b_t)c$.
If we consider the whole financial system with its N banks and following the same methodology described in the second section, the *aggregate vulnerability* is equal to:

$$AV = \frac{1'AMLM' \times [BA_t R_t + (I_N + B_t) \times C]}{E_t + P}$$

4.4.2 Results and analysis

The capital injection simulation show that such a policy is very efficient. In fact, it reduces the *aggregate vulnerability* by 266% and we even find a positive value of it. This is because the central bank, by injecting capitals in banks, reduces their leverage ratio and so, even when the shock happens at the end of the period, the amount of sales will be considerably reduced. [18]

5 Conclusion

Since the outbreak of the financial crisis of 2007 in the United States, regulators are increasingly concerned about the emergence of new systemic crises, which would affect the whole financial system through the contagion phenomenon. The magnitude of the recent crisis has then led several central banks to intervene by adopting non-conventional measures.

In this paper, we introduce, a central bank in the model of contagion proposed by Greenwood et al. (2015), in order to study such policies. A first theoretical framework, based on an asset purchase policy, shows that for two banks owning the same two assets, the central bank should repurchases banks assets to lower systemic risk increases with respect to both bank's leverage ratio and size. We show also, in the case of two banks holding only one asset, that the total budget of the central bank should be allocated to the bank with the highest leverage ratio.

Furthermore, by simulating different unconventional monetary policies, We find that a capital injection measure is the most efficient. However, this is not true for a LTRO policy which leads to an increase of the fire sale effects. Alternatively, the assets purchase policy, for its part, is more effective when the central bank buys specific sovereign debts.

[18]We find here the same rank determined in an asset purchase policy.

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Table 5: Summary statistics for 2011 Stress Tests. We report some descriptive statistics related to 2011 Stress tests. System denotes the sum for all banks assets and equities, Min denotes the minimum value in the sample and Max refer to the maximum value.

	System	Min.	Max.	Mean.
Assets (€billions)	23156	0.339	1444	257
Equity (€billions)	952	0.020	87	11
Leverage	-	3.6	540.8	33.6

Figure 1: Repartition of assets purchase among countries. We represent the aggregate optimal amounts allocated to each country in the case of an asset purchase policy.

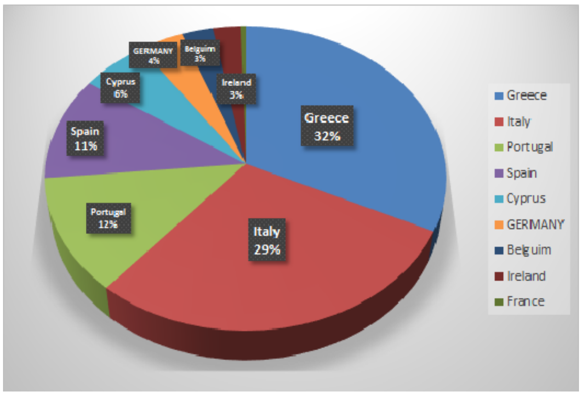

Table 6: Optimal asset purchase policy. We simulate our framework for an asset purchase policy to find an optimal repartition of the €1000 billion. We report here the top 10 banks ranked in order from highest amount allocated by the central bank to the lowest.

Bank	c_i	shock	Leverage	Size(a/E)
BANCA MONTE DEI PASCHI DI SIENA (Italy)	1.9 E+11	-0.055	45.65	0.21
NATIONAL BANK OF GREECE (Greece)	10E+10	-0.115	12.64	0.11
EFG EUROBANK ERGASIAS (Greece)	9.8E+10	0,059	28,25	0.10
BANCO COMERCIAL PORTUGUES (Portugal)	6.7E+10	-0,04	27,15	0.10
BANCO BPI (Portugal)	4.9E+10	-0.055	22,09	0.05
PIRAEUS BANK GROUP (Greece)	4.72E+10	-0.113	16,68	0.05
MARFIN POPULAR BANK PUBLIC (Cyprus)	4.3E+10	0.052	20,27	0.05
INTESA SANPAOLO (Italy)	4.22 E+10	-0.036	21.43	0.61
BANCA MONTE DEI PASCHI DI SIENA (Italy)	3.64 E+10	-0.032	30.84	0.13
CAJA ESPANA DE INVERSIONES (Spain)	3.53 E+10	-0.033	27.38	0.05

Table 7: Difference in assets weights after and before the new optimal assets purchase policy. We report the new values of the weight matrix after the second optimal asset purchase policy for the five top ranked banks: We consider as input the optimal amounts exposed in part in table 5 and propose to determine the asset classes that the central bank should buy to further reduce the *aggregate vulnerability*. To see the intuition, if we consider that the initial weight of Greece sovereign debt in the portfolio of a bank i is equal to 0.5 and after the optimal assets purchase policy this weight becomes equal to 0.2, we report here a value equal to -0.3, i.e the central bank will purchase an amount α of Greece sovereign debt. S.V denotes the sovereign debt. Other assets denotes the aggregate weights of the remained assets composing the banks' portfolio.

Bank	Greece S.D	Ireland S.D	Italy S.D	Portugal S.D	Spain S.D	Other assets
BANCA MONTE ..	0	0	-0,16	-0,001	-0,001	0,16
NATIONAL BANK..	-0,18	0	0	0	0	0,18
EFG EUROBANK..	-0,09	0	-0,001	0	0	0,09
BANCO COMERCIAL..	-0,007	-0,002	-0,0005	-0,07	0	0,08
BANCO BPI..	-0,007	-0,006	-0,02	-0,08	0	0,11

Appendix

1. Two banks and one asset

1.1. Resolution of the maximization problem

If we consider two banks holding one asset and we maximize AV under a budget constraint then our problem is then the following:

$$\begin{aligned} \underset{(c_1,c_2)}{\text{maximize}} \quad & \frac{\gamma_1[\pi_1 + \Sigma_1 c_1] + \gamma_2[\pi_2 + \Sigma_2 c_2]}{E_1 + E_2} \\ \text{subject to} \quad & c_1 + c_2 = P \quad and \quad c_1, c_2 > 0 \end{aligned}$$

We remind that :

$\gamma_1 = \gamma_2 = (a_1 - c_1)l + (a_2 - c_2)l$
$\pi_n = a_n b_n r_n$
$\Sigma_n = 1 - b_n r_n$

AV is a continuous function on a closed bounded interval and so the maximum exist.

Application of Karush–Kuhn–Tucker (KKT) theorem:

Let $h(c_1, c_2) = c_1 + c_2 - P$ (the equality constraint function) and $g_i(c_1, c_2) = c_i$, $\forall i = 1, 2$ (the inequality constraint functions).

- AV, h and g are C^1. So if there is $\bar{x} = (\bar{c_1}, \bar{c_2})$ verifying the above optimization program and if $\nabla h(\bar{x})$ and $\nabla g_k(\bar{x})$ for $k \in J(\bar{x})$ [19] (constraints qualification), then it exist $\lambda,\ \mu_1,\ \mu_2 \in \mathbb{R}$ such as:

$$\begin{cases} \nabla AV(\bar{x}) + \lambda \nabla h(\bar{x}) + \mu_1 \nabla g_1(\bar{x}) + \mu_2 \nabla g_2(\bar{x}) = 0 \\ g_1(\bar{x}),\ g_2(\bar{x}) \geq 0 \\ h(\bar{x}) = 0 \\ \mu_1 g_1(\bar{x}) = 0 \\ \mu_2 g_2(\bar{x}) = 0 \\ \mu_1,\ \mu_2 \geq 0 \end{cases}$$

- Verification of constraints qualification:
$\nabla h(\bar{x}) = (1, 1)$, $\nabla g_1(\bar{x}) = (1, 0)$, $\nabla g_2(\bar{x}) = (0, 1)$.
$J(\bar{x})$ can not be the set $\{1, 2\}$, else we will have $\bar{c_1} = \bar{c_2} = 0$ (however according to our budget constraint $\bar{c_1} + \bar{c_2} = P > 0$). As a result, the vectors family to consider is $\{(1, 1), e^i\}$[20] with $i = 1, 2$.
Clearly this family of vectors is always linearly independent. So the constraints are qualified.

[19] $J(x) = \{j = 1, 2;\ g_j(x) = 0\}$
[20] e^i denotes the vector with a 1 in the ith coordinate and 0's elsewhere.

Once the KKT conditions are satisfied, we resolve the system exposed above. The resolution results in 4 cases:

- **Case 1:** $\mu_1 \neq 0$ and $\mu_2 \neq 0$. This implies that $\bar{c_1} = \bar{c_2} = 0$ which is impossible for the reason mentioned above.
- **Case 2:** $\mu_1 = 0$ and $\mu_2 \neq 0$. This implies that $\bar{c_2} = 0$ and $\bar{c_1} = P$.
  At this stage, we should verify that $\mu_2 > 0$. This condition implies that:
  $$-l(\pi_1 + \pi_2 + \Sigma_2 P) + \gamma\Sigma_2 - \gamma\Sigma_1 = -\mu_2 < 0$$
  $\Rightarrow$
  $$b_1 > b_2$$[21]
- **Case 3:** $\mu_1 \neq 0$ and $\mu_2 = 0$. This implies that $\bar{c_1} = 0$ and $\bar{c_2} = P$.
  At this stage, we should verify that $\mu_1 > 0$. This condition implies that:
  $$-l(\pi_1 + \pi_2 + \Sigma_2 P) + \gamma\Sigma_1 - \gamma\Sigma_2 = -\mu_1 < 0$$
  $\Rightarrow$
  $$b_1 < b_2$$
- **Case 4:** $\mu_1 = 0$ and $\mu_2 = 0$. This implies that :
  $$\begin{cases} -l(\pi_1 + \pi_2) + -l(\Sigma_1 c_1 + \Sigma_2 c_2) + \gamma\Sigma_1 + \lambda = 0 \\ -l(\pi_1 + \pi_2) + -l(\Sigma_1 c_1 + \Sigma_2 c_2) + \gamma\Sigma_2 + \lambda = 0 \\ c_1 + c_2 = P \\ c_1, c_2 \geq 0 \end{cases}$$
  $\Rightarrow$
  $$\Sigma_1 = \Sigma_2 \ \textit{impossible for } b_1 \neq b_2$$

1.2. Function concavity

AV is a twice-differentiable function of 2 variables. Its Hessian at (c_1, c_2) is then:

$$H(c_1, c_2) = \begin{pmatrix} -2l\Sigma_1 & -l(\Sigma_1 + \Sigma_2) \\ -l(\Sigma_1 + \Sigma_2) & -2l\Sigma_2 \end{pmatrix}$$

The determinant of the hessian is then equal to :

$$4l^2\Sigma_1\Sigma_2 - l^2(\Sigma_1 + \Sigma_2)^2 = l^2(\Sigma_1 - \Sigma_2)(\Sigma_2 - \Sigma_1) < 0, \ \forall b_1 \neq b_2$$

Consequently, AV is neither concave nor convex and we can only have corner solutions .

[21] we assume that $a_1 + a_2 > P$

2. Two banks with two assets

2.1. Resolution of the maximization problem

For two banks and two assets, we resolve the same problem exposed above. The only modification concerns the value of γ_n, which is now equal to $\sum_k(\sum_m(a_m - c_m)m_{mk})l_k m_{nk}$.

Here also AV is a continuous function on a closed bounded interval and so the maximum exist and we can apply the KKT theorem for the same reasons exposed in the case above. The resolution of our system results in 4 cases:

- **Case 1:** $\mu_1 \neq 0$ and $\mu_2 \neq 0$. This implies that $\bar{c_1} = \bar{c_2} = 0$ which is impossible for the reason mentioned above.

- **Case 2:** $\mu_1 = 0$ and $\mu_2 \neq 0$. This implies that $\bar{c_2} = 0$ and $\bar{c_1} = P$.
  At this stage, we should verify that $\mu_2 > 0$. This condition implies that:

  $$(t_{12} - t_{11})\pi_1 + (t_{22} - t_{12})\pi_2 + (t_{12} - t_{11})\Sigma_1 P + \gamma_2\Sigma_2 - \gamma_1\Sigma_1 = -\mu_2 < 0$$

  $\Rightarrow$

  $$(t_{12}-2t_{11})\pi_1+((t_{12}-2t_{11})P+a_2t_{12})\Sigma_1+a_1t_{11} < (t_{12}-2t_{22})\pi_2+(a_1t_{12}-t_{12}P)\Sigma_2+a_2t_{22}$$

- **Case 3:** $\mu_1 \neq 0$ and $\mu_2 = 0$. This implies that $\bar{c_1} = 0$ and $\bar{c_2} = P$.
  At this stage, we should verify that $\mu_1 > 0$. This condition implies that:

  $$(t_{11} - t_{12})\pi_1 + (t_{12} - t_{22})\pi_2 + (t_{12} - t_{22})\Sigma_2 P + \gamma_1\Sigma_1 - \gamma_2\Sigma_2 = -\mu_1 < 0$$

  $\Rightarrow$

  $$(t_{12}-2t_{22})\pi_2+((t_{12}-2t_{22})P+a_1t_{12})\Sigma_2+a_2t_{22} < (t_{12}-2t_{11})\pi_1+(a_2t_{12}-t_{12}P)\Sigma_1+a_1t_{11}$$

- **Case 4:** $\mu_1 = 0$ and $\mu_2 = 0$. This implies that :

  $$\begin{cases} t_{11}\pi_1 + t_{12}\pi_2 + t_{11}\Sigma_1 c_1 + t_{12}\Sigma_2 c_2 + \gamma_1\Sigma_1 + \lambda = 0 \\ t_{12}\pi_1 + t_{22}\pi_2 + t_{12}\Sigma_1 c_1 + t_{22}\Sigma_2 c_2 + \gamma_2\Sigma_2 + \lambda = 0 \\ c_1 + c_2 = P \\ c_1, c_2 \geq 0 \end{cases}$$

  The resolution of this system gives the expression of c_1 and c_2:

  $$c_1 = \frac{(t_{12} - t_{11})\pi_1 + (t_{22} - t_{12})\pi_2 + ((2t_{22} - t_{12})\Sigma_2 - t_{12}\Sigma_1)P + (a_1t_{11} + a_2t_{12})\Sigma_1 - (a_1t_{12} + a_2t_{22})\Sigma_2}{2 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]}$$

  $$c_2 = \frac{(t_{11} - t_{12})\pi_1 + (t_{12} - t_{22})\pi_2 + ((2t_{11} - t_{12})\Sigma_1 - t_{12}\Sigma_2)P + (a_1t_{12} + a_2t_{22})\Sigma_2 - (a_1t_{11} + a_2t_{12})\Sigma_1}{2 \times [(t_{11} - t_{12}) * \Sigma_1 + (t_{22} - t_{12}) * \Sigma_2]}$$

  but we should verify that c_1 and c_2 are positive.

2.2. Function concavity

AV is a twice-differentiable function of 2 variables. Its Hessian at (c_1, c_2) is then:

$$H(c_1, c_2) = \begin{pmatrix} 2t_{11}\Sigma_1 & t_{12}(\Sigma_1 + \Sigma_2) \\ t_{12}(\Sigma_1 + \Sigma_2) & 2t_{22}\Sigma_2 \end{pmatrix}$$

The determinant of the matrix is equal to : $4t_{11}t_{22}\Sigma_1\Sigma_2 - t_{12}^2(\Sigma_1+\Sigma_2)^2$. Consequently, AV is concave only if $4t_{11}t_{22}\Sigma_1\Sigma_2 \geq t_{12}^2(\Sigma_1 + \Sigma_2)^2$, because we have already the trace of H which is negative ($2t_{11}\Sigma_1 + 2t_{22}\Sigma_2 \leq 0$ since $tij \leq 0$, $\forall i, j = 1, 2$).