Stop ou Encore: On the contingent corporate decisions to invest and disinvest*

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Abstract

In this paper we consider the valuation of a firm that holds simultaneously an option to expand and to abandon productions depending on the state of the market (good or bad). We investigate the corresponding optimal decision levels in the cases of the infinite and finite time horizon. Analytical formulas for the firm's value are provided. Numerical results document the behavior of the firm's value and optimal decision thresholds. We show how decision thresholds vary with time when the real options are finite-lived. We also show that not considering both opportunities simultaneously can lead to a suboptimal investment/disinvestment decision for the firm.

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1. Introduction

To assess investment decisions, it is standard in economics and management to consider the present value of the expected cumulative future cash flow that will be generated by the project. If the net present value[1] (NPV) is positive then the project can be decided. Often, the underlying assumptions of this popular and simple approach lead to a significant misestimation of the firm's (investment/disinvestment) opportunity. Among the most critical concerns, this valuation technique does not account for the *flexibility*, the firm can have in the management of the project, such as the possibility of delaying investment (or sequentially increasing production) and disinvestment. Such decisions, furthermore, have to be made under uncertainty. These two intricate decisions are typically similar to the management of an option to expand/abandon productions. In real-life situations, the randomness and fluctuations of the cash-flows generated by the project highly influence the decisions made by the managers. In other words, their decisions are clearly contingent. Most often the decisions imply the payment of *sunk costs*[2]. In this paper, we investigate situations where the firm's manager can decide, in the future, to expand (increase firm's capacity) or to abandon their activities. The *real option*[3] approach, we undertake, will take into consideration these important aspects. The real option approach may be viewed as an application of some concepts of financial asset pricing theory to real-life concerns such as project analysis, firm valuation, etc. In fact, the level of cash flow that will be generated by the project is critical for the manager's decision and is assumed to behave as a financial asset. Thus, the real option approach can a) take into account uncertainty of cash flows in the investment environment, b) deal with the contingency of the manager's decision as it depends on the state of the market (favorable/unfavorable) and, c) provide normative insights to decide whether and when to delay the investment/disinvestment decision.

Considering an option to expand or abandon activities may be of paramount importance for the firm. The real option approach was initiated by Black & Scholes (1973) who succeeded in pricing the equity of a levered company as a call option written on the firm's assets value. They state that it is possible to take the equity in a levered company as a call option on the company's value. Later, Merton (1973) extends the work of Black and Scholes with additional assumptions pricing corporate liabilities. Their approach was extended in many directions for analyzing firm concerns such as investment and disinvestment decisions and more. Kester

[1]The net present value is the present value of the expected cumulative future cash flow that will be generated by the project net of the investment cost.

[2]A sunk cost is a cost that cannot be recovered once it is incurred.

[3]A real option is a right and not an obligation to undertake a certain firm's decisions, such as abandoning, expanding, deferring, etc., its activities.

(1984) argues that the opportunity of a firm to invest or expand its activities resembles to the decision to exercise a call option. Trigeorgis (1993) and Copeland, Antikarov, & Copeland (2001) explain that a real option is a right and not an obligation to make an investment decision at a prespecified cost (that may be viewed as the strike) and within a given period of time (the maturity of the option). A firm may also be worth abandoning activities. Myers & Majd (1990) discuss this critical and important decision and analyzed the decision to abandon a project by means of an American put option. The empirical investigation of Kaiser & Stouraitis (2001) evidences the importance, for a firm, of abandoning activities with low value creation. Wong (2006, 2009) analyses the effect of abandonment options on operating leverage. Clark, Rousseau, & Gadad (2010) offer an empirical work on abandonment options. McDonald & Siegel (1986) evaluates the option to differ an investment spending[4]. The book of Dixit & Pindyck (1994) is the turning point for development and application of real options theory in investment decisions under uncertainty. They investigate a wide range of problems solvable by real option in the Black-Scholes' setting. They notably apply the dynamic programming approach for pricing purposes. The value of a project in the real option approach is equal to its value obtained using traditional net present value (NPV) augmented with the value of the flexibility (the value of the real option).

In almost all the aforementioned references, the opportunity to invest and to disinvest decisions are not considered simultaneously. There exist many situations where the firm's manager has to consider the possibility to expand their ongoing activities or abandon them depending on the behavior of future cash flows that will be generated. This is typically the case of start-ups' managers, for example. Dixit & Pindyck (1994) (Chapter 7) address this issue by valuing a firm with a combined Entry-Exit option. They assume that the firm could suspend (exit) operation, and resume (entry) it later without any cost. This assumption appears, however, not that realistic specially in the example, they consider, of a research laboratory engaged in the development of a new pharmaceutical product. Abandoning the project means losing the team of research scientists and hence the ability to resume the project in the future. This may also be the case when it is very costly to pause activities more than undertaking a new project. Some papers have investigated the combined decisions under different scenarios. Costeniuc, Schnetzer, & Taschini (2008) analyses the entry-exit decisions when there is a time lag from the time the decision is taken to time when the decision is implemented. Feil & Musshoff (2013) consider the investment and disinvestment decisions under competition and different market intervention. Kwon (2010) was the first

[4]For more reading about real options and their applications, one may consult Dixit & Pindyck (1994), Smith & Nau (1995), Trigeorgis (1999), Brennan & Trigeorgis (2000), Copeland & Antikarov (2003), Grenadier (2000) and Trigeorgis & Smith (2004) with an application in game theory, Brach (2003), Trigeorgis & Schwartz (2001), Damodaran (2016) among others.

to study the investment decision in the presence of a permanent exit option. He considers a declining market and uses a Brownian motion with negative drift to model the firm's profit stream. Matomäki (2013) uses the geometric Brownian motion and generalizes Kwon (2010) while Hagspiel, Huisman, Kort, & Nunes (2016) extends the works to a firm that also decides on capacity in a declining market.

It's worth noting that most of the literature in real option consider a perpetual time horizon for the investment/disinvestment decision especially those that consider the combined decisions. It must stress that the delay of the decision may sometimes not be perpetual. Temporal constraints are typically imposed by the competitors, technological evolution or regulation constraints, etc. It's also worth investigating different market and firm configurations since the valuation models proposed in the literature does not apply for all firm's.

This paper proposes a new analysis of the close interplay between the decision to expand and to abandon a business. It considers the situation of a firm, with ongoing activities, that holds simultaneously an opportunity to expand and a possibility to abandon its activities. It is assumed that the expansion decision will be undertaken when the present value of the cash flows reaches an upper barrier and that the abandonment decision will occur whenever the present value of the cash flows drops below a lower barrier. Note that, by contrast with Dixit and Pyndick, it is assumed that there is no possibility to resume activities once they are abandoned. This is justified by the example of the research laboratory, discussed earlier, or in some situations where maintaining suspended activities may cost more than investing in a new project. Under these assumptions, we analyze the cases of the infinite and finite expiry time horizon for the decisions and provide analytical formulas for the firm's value based on the dominance arguments. Decision to expand or abandon the project can be taken at any time before or at the maturity. Optimal decision thresholds are then obtained. They are key for the manager's decision and are time-varying when the decision time horizon is finite. Numerical simulations highlight how the value of the firm's opportunity behaves and the way the boundaries are interconnected. They suggest that a manager holding simultaneously the two options may wait longer before deciding to exercise one of the options in comparison with a manager holding only one option (either the abandonment option or the expansion option). The holder of a perpetual option to decide may also wait longer before exercising his option in comparison with the one holding a finite lived option. At the end, we measured the sensitivity of the firm's value with respect to different parameters showing how the firm's value and the manager's decision can drastically change for little variations.

The rest of this paper is organized as follows: Section 2 presents some notations and the valuation framework, Section 3 values the firm's opportunity when the manager has an infinite horizon for decision-making. The section considers three different situations. In the first subsection, it is considered a single option to abandon. In the second subsection, there is only one option to expand. In the third subsection, the manager has both options (the coupled options). Section 4 is organized as Section 3 except that the manager has a finite time horizon for decision-making. Section 5 presents numerical simulations and discussions. Section 6 gives some concluding remarks.

2. The Framework

We consider the McDonald & Siegel (1986)'s pricing model and assume that the underlying asset P, which is the cash flow generated by the project, follows a geometric Brownian motion defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q}\right)$ where $\mathcal{F}_t$ represents the information available at time t. The cash flow[5] process is described by

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad t \geq 0, \quad P_0 \quad \text{given,} \tag{1}$$

where W is a standard Brownian motion, μ is the trend or the expected rate of return of the project, and σ the volatility of the project cash flows. Parameters μ and σ are supposed to be constant. Unlike financial assets, real assets are not traded on financial markets[6]. Consequently the trend of the project may differ from the expected rate of return in equilibrium in financial markets.

We denote by V the value of the firm. This value is a contingent or derivative asset, whose payoff depends on the value of the underlying asset P. If future revenues are discounted at the rate r, it is well known (see for instance Merton (1973)) that the value $V(P_t, t)$ of every contract written on P satisfies the following fundamental partial differential equation

$$\frac{\partial V}{\partial t} + (r - \delta) P \frac{\partial V}{\partial P} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 V}{\partial P^2} - rV = 0, \quad 0 \leq t \leq T. \tag{2}$$

The constant $\delta = r - \mu$ acts as a dividend payment rate on the project value.

Our starting point is the value of a firm with no real option and generating infinitely cash flows driven by the process $(P_t)_{t\geq 0}$. Denote by $\widetilde{V}(P_t, t)$ the value of such a firm, i.e., the expected present value of cash flows that will be generated, at any time t. If $E_t[\cdot] \equiv E[\cdot \mid \mathcal{F}_t]$ denotes the conditional expectation relative to $\mathbb{P}$ given information $\mathcal{F}_t$ then the following proposition holds.

[5] For simplicity, the cash flows are considered net of the production costs, and this, without lost of generality unless the production costs are suspected of being stochastic.

[6] The volatility of the project cash flows can be estimated from simulations or historical data of publicly traded firms specialized in the considered business.

Proposition 1 (Dixit & Pindyck (1994)). *Let $(P_t)_{t \geq 0}$ be a geometric Brownian motion following Equation (1). If the risk-free interest rate $r > \mu$ then the expected present value, at any time t, of the cumulative cash flow driven by the price process $(P_t)_{t \geq 0}$ is given by*

$$\widetilde{V}(P_t, t) = E_t\left[\int_t^{\infty} P_s e^{-r(s-t)} ds\right] = \frac{P_t}{r - \mu} \tag{3}$$

where $E_t[.]$ is the mathematical expectation under risk-neutral probability and given P_t.

Proposition 1 is a well-known and important result of real option analysis. Dixit & Pindyck (1994) designate $\widetilde{V}(P_t, t)$ as the "fundamental component".

Proof of Proposition 1. Solving Equation (1) yields to

$$P_s = P_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(s-t) + \sigma(W_s - W_t)},$$

and for $s \geq t$, we have $E_t[P_s] = P_t e^{\mu(s-t)}$. Thus, using the Guido Fubini's theorem, the expected present value of cash flows is

$$\begin{aligned} E_t\left[\int_t^{\infty} P_s e^{-r(s-t)} ds\right] &= \int_t^{\infty} E_t[P_s] e^{-r(s-t)} ds \\ &= \int_t^{\infty} P_t e^{-(r-\mu)(s-t)} ds \\ &= \frac{P_t}{r - \mu}. \end{aligned}$$

□

3. Investment decision under an infinite time horizon

The literature on real options most often considers an infinite time horizon for investment/disinvestment decisions. In this section, we present general results useful for valuing firms with additional flexibility in decisions. Following the standard approach, the firm's manager is supposed to have infinite time horizon for the decisions.

When considering infinite time horizon, the derivative with respect to the time t, in Equation (2), vanishes and we have

$$\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 V}{\partial P^2} + (r - \delta) P \frac{\partial V}{\partial P} - rV = 0. \tag{4}$$

This is the PDE to consider for deriving analytical expressions for "perpetual" real options.

3.1. *Decision to abandon*

Under **unfavorable** market conditions, i.e., when the value of cash flow drops below a lower exogenous level L (at the stopping time τ_L), the manager of the firm can decide to abandon the production. Note that abandoning activities at τ_L implies losing the cash flows that would be generated if activities continue beyond τ_L. The decision threshold L is time-invariant due to the absence of a time constraint, that is, the manager has an infinite time horizon for the decision. We assume that, when the abandonment is decided, the manager receives the salvage value B[7]. Let τ_L be the first hitting time of a constant exogenous level L and B the salvage value of the firm. The rational value of such a firm is given by the following proposition:

Proposition 2. *The value of the firm with an option to abandon, $V_L^{ab}(P_t, t)$, is given by*

$$V_L^{ab}(P_t, t) = E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds + B e^{-r(\tau_L - t)}\right] \tag{5}$$

where $\tau_L \equiv \inf\{\varepsilon \geq t : P_\varepsilon = L\}$, or $\tau_L \equiv \infty$ if no such time exists in $[t, \infty)$. $V_L^{ab}(P_t, t)$ has the analytical expression as follows:

$$V_L^{ab}(P_t, t) = \frac{P_t}{r-\mu} + \left(B - \frac{L}{r-\mu}\right)\left(\frac{P_t}{L}\right)^{\theta_0} \tag{6}$$

where

$$\theta_0 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2} < 0$$

with

$$0 < L \leq (r-\mu) B.$$

It is understood from Equation (5) that the project will generate cash flows from time t until the random abandonment date τ_L and that the manager receives the salvage value at that time (B). If ever the level L is never reached from $t \geq 0$, then the first hitting time is $\tau_L \equiv \infty$ and the second component $Be^{-r(\tau_L - t)}$ collapses. In such a case, $V_L^{ab}(P_t, t)$ is simply equal to the value of the fundamental component.

[7] The salvage value is the estimated value of the project upon its sale at the abandonment date. It can result from the sale of machines and furniture, for example.

The expression in (6) decomposes the value of the firm as the sum of the fundamental component and the value of the option to abandon. Recall that the level L is considered exogenous in this context. This property is important for examining the optimal decision level that will be discussed later. The additional condition (inequalities in the last line of the above proposition) ensures the rationality of the financial decision. The option to abandon must be indeed positive. So, we must have $(B - L/(r-\mu)) \geq 0$.

Proof of Proposition 2. To demonstrate the proposition we will analyze both cases that arise for the firm.

Case 1: The stopping time τ_L exists in $[t, \infty)$

The activities of the firm will generate cash flows driven by the process $(P_t)_{t\geq 0}$ until the abandonment date τ_L. The expected cumulative discounted cash flows is represented as follows: $E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds\right]$. At that stopping date τ_L the salvage value of the firm is assumed to be equal to B. We end up with

$$V_L^{ab}(P_t, t) = E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds + B e^{-r(\tau_L - t)}\right].$$

Case 2: The stopping time τ_L does not exist in $[t, \infty)$

In that case, $\tau_L \equiv \infty$, and this situation is similar to that of a firm with no option to abandon activities. The project will infinitely generate cash flows from time t. Thus $V_L^{ab}(P_t, t)$ is equal to the expected present value of cash flows generated and the discounted value of B is just zero for the infinite time horizon. The value of the firm is given by the fundamental component

$$V_L^{ab}(P_t, t) = E_t\left[\int_t^{\infty} P_s e^{-r(s-t)} ds\right].$$

Thanks to some well-known properties of Brownian motion, the analytical expression can be derived for $V_L^{ab}(P_t, t)$. The expectation in Equation (5) is decomposed into two components. The first is the expected present value of the cumulative cash flow generated until the random abandonment time τ_L and it is equal to

$$E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds\right].$$

The second is the expected present value of the salvage value $E_t\left[B e^{-r(\tau_L - t)}\right]$. Its value is computed from the Laplace transform of τ_L (see Karatzas & Shreve (1998), p.63–67) and it is given by the solution of the homogeneous part of the Equation (4):

$$E_t\left[e^{-r(\tau_L-t)}\right]=\left(\frac{P_t}{L}\right)^{\theta_0}$$

where

$$\theta_0=\frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2}<0.$$

An alternative approach is to solve the PDE and then view θ_0 as the negative root of the characteristic fundamental quadratic equation

$$\frac{1}{2}\sigma^2\theta\left(\theta-1\right)+\left(r-\delta\right)\theta-r=0. \tag{7}$$

Once the expectation $E_t\left[Be^{-r(\tau_L-t)}\right]$ is obtained, we can compute $V_L^{ab}\left(P_t,t\right)$ and we have

$$\begin{aligned}
V_L^{ab}\left(P_t,t\right) &= E_t\left[\int_t^{\tau_L}P_se^{-r(s-t)}ds\right]+E_t\left[Be^{-r(\tau_L-t)}\right]\\
&= E_t\left[\int_t^{\infty}P_se^{-r(s-t)}ds\right]\\
&\quad -E_t\left[e^{-r(\tau_L-t)}E_{\tau_L}\left[\int_{\tau_L}^{\infty}P_se^{-r(s-\tau_L)}ds\right]\right]+E_t\left[Be^{-r(\tau_L-t)}\right]\\
&= E_t\left[\int_t^{\infty}P_se^{-r(s-t)}ds\right]\\
&\quad +E_t\left[\left(B-E_{\tau_L}\left[\int_{\tau_L}^{\infty}P_se^{-r(s-\tau_L)}ds\right]\right)e^{-r(\tau_L-t)}\right]\\
&= E_t\left[\int_t^{\infty}P_se^{-r(s-t)}ds\right]\\
&\quad +E_t\left[\left(B-E_{\tau_L}\left[\int_{\tau_L}^{\infty}P_se^{-r(s-\tau_L)}ds\right]\right)\right]\times E_t\left[e^{-r(\tau_L-t)}\right]\\
&= \frac{P_t}{r-\mu}+\left(B-\frac{L}{r-\mu}\right)\left(\frac{P_t}{L}\right)^{\theta_0}.
\end{aligned}$$

Note that

$$\begin{aligned}
&E_t\left[\left(B-E_{\tau_L}\left[\int_{\tau_L}^{\infty}P_se^{-r(s-\tau_L)}ds\right]\right)e^{-r(\tau_L-t)}\right]\\
&\qquad =E_t\left[\left(B-E_{\tau_L}\left[\int_{\tau_L}^{\infty}P_se^{-r(s-\tau_L)}ds\right]\right)\right]\times E_t\left[e^{-r(\tau_L-t)}\right].
\end{aligned}$$

In fact, the variables τ_L and P_{τ_L} are independent. □

Because this decision is strategic, the manager may endogenize the decision threshold L, i.e. choose L so as to maximize the firm's value. Let's denote by L^* the optimal level that maximizes $V_L^{ab}(P_t,t)$ over L. We have the following properties for the function $V_L^{ab}(P_t,t)$:

Theorem 3.1. *Let $V_L^{ab}(P_t,t)$ be the value of a firm with an option to abandon activities at level L. We have*
i)$V_L^{ab}(P_t,t)$ is continuous on $\mathbb{R}_+^2$,
ii) the inequality

$$V_L^{ab}(P,t) \leq V_{L^*}^{ab}(P,t), \quad \forall P > 0, \quad L > 0$$

where

$$L^* = \frac{\theta_0}{\theta_0 - 1}(r-\mu)\, B.$$

The continuity argument in i) follows from the properties of payoff functions of the option side[8] in Equation (6). Property ii) provides the expression of the optimal level L^* at which the manager can optimally decide to abandon the activities.

Proof of Theorem 3.1. The property i) is ensured by the continuity of the option payoff function and the continuity of the cash flows process driven by Equation (1).

To prove ii) we define the function $\phi(L) = (B - L/(r-\mu))/L^{\theta_0}$. This function is an increasing function on $(0, L^*)$ and decreasing on (L^*, ∞). Consequently it has its maximum on $(0,\infty)$ at L^*. □

The value of the optimal level L^* can be plugged into Equation (6) to obtain

$$\widetilde{V}_*^{ab}(P_t,t) = \frac{P_t}{r-\mu} + \frac{B}{1-\theta_0}\left(\frac{P_t/(r-\mu)}{B\ \theta_0/(\theta_0-1)}\right)^{\theta_0}. \tag{8}$$

3.2. Decision to expand

Under **favorable** market conditions, the manager of the firm can decide to expand current activities by a factor α. Here, it is assumed that the firm is compelled to continue activities despite unfavorable conditions. This is realistic for some public services or when the cash flows level is significantly high so that the probability of falling below a defined barrier becomes negligible. The decision to expand will be at an optimal stopping time when the present value of the cash flow reaches or exceeds an upper barrier H. We also assume that the manager will face a constant investment cost I_α[9] when increasing the firm's capacity by α.

[8] Here the option side is the component of the equation beside the fundamental component $P_t/(r-\mu)$.

[9] We let I_α unspecified for the seek of generality. Obvious specifications are linear investment costs so that $I_\alpha = \alpha I$ where I is the cost of one unit of investment.

Let τ_H be the first hitting time of an exogenous level H and I_α the cost of investment of increasing the firm's capacity by α. The next proposition is concerned with the value of a firm with an option to expand its activities.

Proposition 3. *The rational value of the firm with an option to expand , $V_H^{exp}(P_t, t)$, is given by*

$$V_H^{exp}(P_t, t) = E_t\left[\int_t^\infty P_s e^{-r(s-t)} ds\right] + E_t\left[e^{-r(\tau_H - t)}\left(\alpha E_{\tau_H}\left[\int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)} ds\right] - I_\alpha\right)\right] \quad (9)$$

where $\tau_H \equiv \inf\{\varepsilon \geq t : P_\varepsilon = H\}$, or $\tau_H \equiv \infty$ if no such time exists in $[t, \infty)$. The value $V_H^{exp}(P_t, t)$ has the analytical expression as follows:

$$V_H^{exp}(P_t, t) = \frac{P_t}{r-\mu} + \left(\frac{\alpha H}{r-\mu} - I_\alpha\right)\left(\frac{P_t}{H}\right)^{\theta_1} \quad (10)$$

where

$$\theta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2} > 1$$

with

$$H \geq (r-\mu) I_\alpha / \alpha.$$

The intuition behind Equation (9) is that a firm with an option to expand activities has at least the cash flows that will be infinitely generated by the initial project (first expectation). If the option is exercised then additional cash flows times α are obtained given an investment cost I_α (second expectation). Otherwise $V_H^{exp}(P_t, t)$ is no more than the fundamental component.

The expression of Equation (10) also decomposes the value of the firm into the sum of the fundamental component and the value of the option to expand. Later, we will see that it is important for the parameter θ_1 to be greater than 1 for the determination of an optimal decision level. The final condition $H \geq (r-\mu) I_\alpha / \alpha$ is also imposed for the seek of rationality as explained in Proposition 2.

Proof of Proposition 3. To prove the proposition, let us analyze the two scenarios as in the proof of Proposition 2.

Case 1: The stopping time τ_H exists in $[t, \infty)$

The activities of the firm will generate cash flows driven by the process $(P_t)_{t\geq 0}$ until the random abandonment date τ_H. The random discounted cumulative cash flow is represented as follows:

$$\int_t^{\tau_H} P_s e^{-r(s-t)} ds.$$

At that stopping time τ_H, the present value of the future cumulative cash flow of the new production capacity, from τ_H, is

$$(1+\alpha) \int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds.$$

The new investment cost, I_α, is deducted from the global revenue. Thus the expected present value of the firm is

$$V_H^{exp}(P_t, t) = E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds \right. \\ \left. + \left(\alpha E_{\tau_H} \left[\int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds \right] - I_\alpha \right) e^{-r(\tau_H - t)} \right]. \quad (11)$$

Case 2: The stopping time τ_H does not exist in $[t, \infty)$
It means that $\tau_L \equiv \infty$ and we have the particular case of Equation (11). The project will infinitely generate cash flows from time t. $V_H^{exp}(P_t, t)$ is therefore equal to the expected present value of the cumulative cash flow that will be generated as the discounted value of I_α is just zero for the infinite time horizon. The value of the firm is given by

$$V_H^{exp}(P_t, t) = E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds \right].$$

A closed-form formula is available for the value of the firm. Recall that the value of the firm is the sum of the present value of cumulative cash flow of the initial investment

$$E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds \right] = \frac{P_t}{r-\mu}$$

and the expected present value of cumulative cash flow of the newly added capacity from τ_H to infinity:

$$\alpha e^{-r(\tau_H - t)} E_{\tau_H} \left[\int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds \right].$$

The sum is reduced by the present value of the investment cost I_α. Summing up, we have

$$\begin{aligned} V_H^{exp}(P_t,t) &= E_t\left[\int_t^\infty P_s e^{-r(s-t)}ds\right] \\ &\quad + E_t\left[\alpha e^{-r(\tau_H-t)}E_{\tau_H}\left[\int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)}ds\right]\right] - E_t\left[I_\alpha e^{-r(\tau_H-t)}\right] \\ &= \frac{P_t}{r-\mu} + E_t\left[e^{-r(\tau_H-t)}\left(\alpha E_{\tau_H}\left[\int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)}ds\right] - I_\alpha\right)\right] \\ &= \frac{P_t}{r-\mu} + E_t\left[\left(\alpha E_{\tau_H}\left[\int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)}ds\right] - I_\alpha\right)\right] \times E_t\left[e^{-r(\tau_H-t)}\right] \\ &= \frac{P_t}{r-\mu} + \left(\frac{\alpha H}{r-\mu} - I_\alpha\right)\left(\frac{P_t}{H}\right)^{\theta_1}. \end{aligned}$$

where

$$E_t\left[e^{-r(\tau_H-t)}\right] = \left(\frac{P_t}{H}\right)^{\theta_1}$$

with θ_1 the second root of Equation (7).

In addition $\alpha H/(r-\mu) - I_\alpha$ must be positive. Thus we have

$$H \geq (r-\mu)\, I_\alpha/\alpha.$$

□

The manager can also endogenize the decision level H in order to maximize the value of the firm. An optimal decision threshold H^* can be derived by maximizing $V_H^{exp}(P_t,t)$ over H. This is given in next theorem.

Theorem 3.2. *Let $V_H^{exp}(P_t,t)$ be the value of a firm with an option to expand activity by α at level H. We have*
i)$V_H^{exp}(P_t,t)$ is continuous on $\mathbb{R}_+^2$,
ii) the inequality

$$V_H^{exp}(P_t,t) \leq V_{H^*}^{exp}(P_t,t), \quad \forall P>0, \quad H>0$$

where

$$H^* = \frac{\theta_1}{\theta_1-1}\frac{r-\mu}{\alpha}I_\alpha.$$

The continuity in i) is a consequence of the continuity of the option's payoff functions and the continuity of cash flows process. Property ii) provides the value of the optimal level H^* for the manager to decide for the expansion of the activity.

Proof of Theorem 3.2. i) The continuity argument developed in Theorem (3.1) also holds for the function $V_H^{\exp}(P_t, t)$.

For the proof of ii) we consider the function $\phi(H) = (\alpha H/(r-\mu) - I_\alpha) H^{-\theta_1}$. This latter is an increasing function on $(0, H^*)$, decreasing on (H^*, ∞), and thus has it maximum on $(0, \infty)$ at H^*. □

The expected maximum value of the firm is obtained by plugging H^* into Equation (10) and we have

$$V_*^{exp}(P_t, t) = \frac{P_t}{r-\mu} + \frac{I_\alpha}{\theta_1 - 1}\left(\frac{\alpha P_t/(r-\mu)}{I_\alpha\ \theta_1/(\theta_1 - 1)}\right)^{\theta_1}.$$

3.3. *Combining abandonment and expansion decisions*

In previous subsections, the decision to abandon and to expand are analyzed independently. What if the manager has both possibilities for the firm? Expand the activities when market conditions are favorable or abandon under unfavorable market conditions. This is the case of many start-ups and firms with ongoing activities.

To value such firms, it is assumed that managers will take the decisions at exogenous thresholds; expand at H and abandon at L. Recall that in our context there is no possibility to resume activities once it is abandoned. The cash flow process $(P_t)_{t\geq 0}$ should satisfy the inequalities $0 < L < P_t < H$ at the inception date t. Two cases arise depending on whether the upper barrier H is first to be reached, by $(P_t)_{t\geq 0}$, or the lower barrier L. If the upper barrier H is reached first at time τ_H (case 1), activities will be extended by α, given an investment cost I_α and the global activity will continue generating the cumulative cash flow. If the lower barrier L is reached first at time τ_L (case 2), the cash flows are interrupted at τ_L and we have a salvage value of B. As explained previously, we assumed that once the project is abandoned there is no possibility to extend activities in the future. This is summarized as follows:

Proposition 4. *Let τ_L and τ_H be respectively the first hitting time of exogenous levels L and H. For $0 < L < P_t < H$ and $t \geq 0$, the value of the firm with an option to abandon and an option to expand activities by α, $V_{L,H}^c(P_t, t)$,is given by*

$$V_{L,H}^c(P_t, t) = E_t\left[\Phi_{t,1} 1_{\{\tau_H < \tau_L\}} + \Phi_{t,2} 1_{\{\tau_L < \tau_H\}}\right] \tag{12}$$

where

$$\Phi_{t,1} = \int_t^\infty P_s e^{-r(s-t)} ds + \alpha e^{-r(\tau_H - t)} E_{\tau_H}\left[\int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)} ds\right] - I_\alpha e^{-r(\tau_H - t)}$$

and

$$\Phi_{t,2} = \int_t^{\tau_L} P_s e^{-r(s-t)} ds + B e^{-r(\tau_L - t)}.$$

The value $V^c_{L,H}(P_t, t)$ has the analytical expression as follows:

$$V^c_{L,H}(P_t, t) = \frac{P_t}{r-\mu} + \left(B - \frac{L}{r-\mu}\right) \times \frac{\left(\frac{P_t}{H}\right)^{\theta_0} - \left(\frac{P_t}{H}\right)^{\theta_1}}{\left(\frac{H}{L}\right)^{-\theta_0} - \left(\frac{H}{L}\right)^{-\theta_1}} + \left(\frac{\alpha H}{r-\mu} - I_\alpha\right) \times \frac{\left(\frac{P_t}{L}\right)^{\theta_0} - \left(\frac{P_t}{L}\right)^{\theta_1}}{\left(\frac{H}{L}\right)^{\theta_0} - \left(\frac{H}{L}\right)^{\theta_1}}. \quad (13)$$

Each of the component $\Phi_{t,1}$ and $\Phi_{t,2}$ represents a particular case depending on the first hitting time of the thresholds. Equation (12) shows how interconnected both decisions are. The equation can be rewritten as

$$\begin{aligned} V^c_{L,H}(P_t, t) &= E_t\left[\Phi_{t,1} 1_{\{\tau_H < \tau_L\}} + \Phi_{t,2} 1_{\{\tau_L < \tau_H\}}\right] \\ &= E_t\left[\Phi_{t,1} \mid \tau_H < \tau_L\right] \times Q(\tau_H < \tau_L) \\ &\quad + E_t\left[\Phi_{t,2} \mid \tau_L < \tau_H\right] \times Q(\tau_L < \tau_H) \end{aligned}$$

where

$$Q(\tau_H < \tau_L) = \frac{\left(\frac{P_t}{L}\right)^{-(\theta_0+\theta_1)} - 1}{\left(\frac{P_t}{L}\right)^{-(\theta_0+\theta_1)} - \left(\frac{P_t}{H}\right)^{-(\theta_0+\theta_1)}}$$

and

$$\begin{aligned} Q(\tau_L < \tau_H) &= 1 - Q(\tau_H < \tau_L) \\ &= \frac{1 - \left(\frac{P_t}{H}\right)^{-(\theta_0+\theta_1)}}{\left(\frac{P_t}{L}\right)^{-(\theta_0+\theta_1)} - \left(\frac{P_t}{H}\right)^{-(\theta_0+\theta_1)}} \end{aligned}$$

are respectively the probability of hitting the upper barrier H first and the probability of hitting the lower barrier L first. Their expressions are derived in Appendix A.

Equation (13) gives closed-form expression for the value of a firm with an option to abandon and to expand activities. This analytical expression is very interesting since it shows that both options, to expand and to abandon, interact and are not simply the sum of the values of two independent real options.

Proof of Proposition 4. To prove the proposition let us analyze both cases.

Case 1: The upper barrier H is reached first $(\tau_H < \tau_L)$

If the upper barrier H is reached first, activities will be first expanded (by α at a given investment cost I_α) and then the cumulative activity of size $(1+\alpha)$ infinitely generates cash flows. The present value of cumulative discounted cash flow of the activity $\int_t^\infty P_s e^{-r(s-t)}ds$ is then augmented by the α unites of the discounted revenue that will be generated by the activity starting at τ_H, that is $\alpha e^{-r(\tau_H - t)} \int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)}ds$, net of the discounted investment cost $I_\alpha e^{-r(\tau_H - t)}$. Summing up, we have

$$\Phi_{t,1} = \int_t^\infty P_s e^{-r(s-t)}ds + \alpha e^{-r(\tau_H - t)} \int_{\tau_H}^\infty P_s e^{-r(s-\tau_H)}ds - I_\alpha e^{-r(\tau_H - t)}.$$

Case 2: The lower barrier L is reached first ($\tau_L < \tau_H$)
If the lower barrier L is reached first, the activities are stopped at time τ_L and we have the salvage value B. It is assumed that there is no possibility to resume the activities in the future once stopped. Otherwise it is evaluated as an independent new project. The (random) firm's value under this scenario is

$$\Phi_{t,2} = \int_t^{\tau_L} P_s e^{-r(s-t)}ds + Be^{-r(\tau_L - t)}.$$

The first result of Proposition (4) follows.

The derivation of the analytical expression is exposed in Appendix B. □

The barriers H and L were considered exogenous. They can, however, be determined optimally so as to maximize the value of the firm. In addition both barriers depend on each other and must be computed numerically. It is obvious that the optimal values of H and L, respectively denoted by L^* and H^*, satisfy the equations

$$\frac{\partial V^c_{L,H}}{\partial L} |_{L=L^*,H=H^*} = 0, \qquad \frac{\partial V^c_{L,H}}{\partial H} |_{L=L^*,H=H^*} = 0.$$

Theorem 3.3. *Let $V^c_{L,H}(P_t, t)$ be the value of a firm with an option to abandon at L and expand activities by α at level H. The function $V^c_{L,H}(P_t, t)$ is continuous on $\mathbb{R}^2_+$.*

This continuity property suggests that the immediate exercise region is a closed set. Furthermore, and by definition, the region above H^* is up-connected and the region below L^* is down-connected. The uniqueness of H^* and L^* then follows.

Proof of Theorem 3.3. As with the functions $V^{ab}_L(P_t, t)$ and $V^{exp}_H(P_t, t)$, the continuity of the function $V^c_{L,H}(P_t, t)$ holds due to the continuity of the option payoff function and the continuity of the cash flows process driven by Equation (1). □

4. Investment decision under finite time horizon

Previously we have considered an infinite time horizon for the financial decisions. In some circumstances, the firm's manager has some time constraint for their decisions to abandon or to expand their activities. This section considers finite lived opportunities to invest/disinvest before or at a given expiry time.

Consider the value of a firm, $\overline{V}(P_t, t)$[10], with a finite lived option written on the underlying process P. The value of the firm also satisfies the PDE of Equation (2). Due to the time constraint, the optimal decision levels are now time-variant. They can be decreasing or increasing functions of time, depending on the nature of the contract (abandonment or expansion). In fact, as time passes and tends to the time horizon, the set of early exercise possibilities shrinks. The following subsections undertake the valuation of firms enjoying different real option opportunities.

4.1. Decision to Abandon

This subsection is about the valuation, at time t, of a firm with an option to abandon activities given that, if necessary, the decision should be taken within a period of time delimited by the interval $[0, T]$. Let us denote by $L(\cdot)$ the unknown time-varying optimal exercise boundary (or early exercise boundary) to determine, and by $\mathcal{S}([t, T]) \equiv \mathcal{S}_{t,T}$ the set of stopping times of the filtration taking values between t and T. The value of the firm with the abandonment option is given by the following proposition.

Proposition 5. *Consider a firm with an option to abandon activities with a salvage value of B. The rational value of the firm , at time t, is given by*

$$\overline{V}_L^{ab}(P_t, t) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t \Bigg[\int_t^{\infty} P_s e^{-r(s-t)} ds + e^{-r(\tau - t)} \left(B - E_\tau \left[\int_\tau^{\infty} P_s e^{-r(s-\tau)} ds \right] \right)^+ \Bigg] \quad (14)$$

The optimal exercise policy is characterized by

$$\tau_t = \inf \{ s \in [t, T] : P_s = L(s) \}.$$

[10]The overline over V indicates that the firm's manager has a finite time horizon for the financial decision.

The value $\overline{V}_L^{ab}(P_t, t)$ *has the analytical expression as follows:*

$$\overline{V}_L^{ab}(P_t, t) = \frac{P_t}{r-\mu} + \frac{1}{r-\mu} P(P_t, t, (r-\mu)B, L(\cdot)) \tag{15}$$

for $t \in [0, T]$*, where* $P(P_t, t, K_B, L(\cdot))$ *is the value of a standard American put option on* P_t *with strike* $K_B = (r-\mu)B$ *and maturity* T*. The early exercise boundary* L *solves the non-linear recursive integral equation*

$$K_B - L(t) = P(L(t), t, K_B, L(\cdot)) \tag{16}$$

for all $t \in [0, T]$*, subject to the boundary condition* $\lim_{t \uparrow T} L(t) = \min\{K_B, (r/\delta) K_B\}$*. At maturity* $L(T) = (r-\mu)B$*.*

This proposition gives a representation of the value of the firm. This representation suggests that the American-style contingent claim ought to equalize the expected maximum value of the global payoff.

Following the EEP representation, the value of a standard American put option can be rewritten as

$$P(P_t, t, K, L(\cdot)) = p(P_t, t) + \pi_p(P_t, t; L(\cdot)), \quad \text{for} \quad 0 \le t \le T, \tag{17}$$

where $p(P_t, t)$ represents the value of a standard European put option given by

$$p(P_t, t) = Ke^{-r(T-t)} \aleph(-d_2(P_t, T-t; K)) - Se^{-\delta(T-t)} \aleph(-d_1(P_t, T-t; K))$$

and $\pi_p(P_t, t; L(\cdot))$ represents the early exercise premium defined by

$$\pi_p(P_t, t; L(\cdot)) = \int_t^T \Big[rKe^{-r(\eta-t)} \aleph(-d_2(P_t, \eta-t; L(\eta))) \\ - \delta P_t e^{-\delta(\eta-t)} \aleph(-d_1(P_t, \eta-t; L(\eta))) \Big] d\eta$$

with

$$d_1(P_t, s-t; \beta) = \frac{\ln(P_t/\beta) + (r-\delta+1/2\sigma^2)(s-t)}{\sigma\sqrt{s-t}},$$
$$d_2(P_t, s-t; \beta) = d_1(P_t, s-t; \beta) - \sigma\sqrt{s-t}.$$

Proof of Proposition 5. The result of the proposition is obtained by analyzing the resulting payoff of the option to abandon. At the inception date t, the firm worth the present value of the cumulative future cash flows

$$\int_t^{\infty} P_s e^{-r(s-t)} ds,$$

and if the manager exercises the option at a date τ within $[t, T]$, He/She will lose the expected present value of cumulative future cash flow from τ onwards and, in return, will receive the salvage value of the firm. Of course, a rational manager will exercise the option only if the salvage value is higher than the expected present value of cumulative future cash flow. The global payoff is given by

$$\int_t^{\infty} P_s e^{-r(s-t)} ds + e^{-r(\tau-t)} \left(B - E_\tau \left[\int_\tau^{\infty} P_s e^{-r(s-\tau)} ds\right]\right)^+ .$$

We end up with the result by applying, to the payoff, the Theorem 5.3 of Karatzas & Shreve (1998).

The analytical expression can be established by considering Equation (14) and we have

$$\begin{aligned}
\overline{V}_L^{\text{ab}}(P_t, t) &= \sup_{\tau \in \mathcal{S}_{t,T}} E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds + e^{-r(\tau-t)} \left(B - E_\tau \left[\int_\tau^{\infty} P_s e^{-r(s-\tau)} ds\right]\right)^+\right] \\
&= E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds\right] \\
&\quad + \sup_{\tau \in \mathcal{S}_{t,T}} E_t \left[e^{-r(\tau-t)} \left(B - E_\tau \left[\int_\tau^{\infty} P_s e^{-r(s-\tau)} ds\right]\right)^+\right] \\
&= \frac{P_t}{r-\mu} + \sup_{\tau \in \mathcal{S}_{t,T}} E_t \left[e^{-r(\tau-t)} \left(B - \frac{P_\tau}{r-\mu}\right)^+\right] \\
&= \frac{P_t}{r-\mu} + \frac{1}{r-\mu} \times \sup_{\tau \in \mathcal{S}_{t,T}} E_t \left[e^{-r(\tau-t)} \left((r-\mu) B - P_\tau\right)^+\right].
\end{aligned}$$

One just has to remark that the maximization of the expectation

$$E_t \left[\int_t^{\infty} P_s e^{-r(s-t)} ds + e^{-r(\tau-t)} \left(B - E_\tau \left[\int_\tau^{\infty} P_s e^{-r(s-\tau)} ds\right]\right)^+\right]$$

over τ, in time interval $[t, T]$, is equivalent to the maximization of the expectation

$$E_t \left[e^{-r(\tau-t)} \left((r-\mu) B - P_\tau\right)^+\right] \tag{18}$$

over τ, in $[t, T]$. Obviously, the last expectation corresponds to the price of a standard American put option on the underlying asset P_t with strike $(r-\mu) B$ and maturity T. □

4.2. Decision to Expand

Now it is considered a firm with a single option to expand its activities by α under favorable market conditions where the manager has a given time interval, $[0,T]$, for the expansion decision. The valuation of the firm is made at an inception time $t \in [0,T]$. Let $H(\cdot)$ be the unknown time-varying EEB and define the optimal expansion time by

$$\tau_t = \inf\{s \in [t,T] : P_s = H(s)\}.$$

The next proposition gives the value of the considered firm.

Proposition 6. *Consider a firm with an option to expand activities by α unite given an investment cost I_α. The rational value of the firm , at time t, is given by*

$$\overline{V}_H^{exp}(P_t,t) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t\left[\int_t^\infty P_s e^{-r(s-t)}ds \right.$$
$$\left. + e^{-r(\tau-t)}\left(\alpha E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)}ds\right] - I_\alpha\right)^+\right]. \quad (19)$$

The optimal exercise policy is given by

$$\tau_t = \inf\{s \in [t,T] : P_s = H(s)\}.$$

The value $\overline{V}_H^{exp}(P_t,t)$ has the analytical expression as follows:

$$\overline{V}_H^{exp}(P_t,t) = \frac{P_t}{r-\mu} + \frac{\alpha}{r-\mu}C(P_t,t,(r-\mu)I_\alpha/\alpha, H(\cdot)) \quad (20)$$

where $C(P_t,t,K_I^\alpha,H(\cdot))$ is the value of a standard American call option on P_t with strike $K_I^\alpha = (r-\mu)I_\alpha/\alpha$ and maturity T. The early exercise boundary H solves the non-linear recursive integral equation

$$H(t) - K_I^\alpha = C(H(t),t,K_I^\alpha,H(\cdot)) \quad (21)$$

for all $t \in [0,T]$, subject to the boundary condition $\lim_{t\uparrow T} H(t) = \max\{K_I^\alpha,(r/\delta)K_I^\alpha\}$. At maturity $H(T) = (r-\mu)I_\alpha/\alpha$.

The value of a standard American call option is given by

$$C(P_t,t,K,H(\cdot)) = c(P_t,t) + \pi_c(P_t,t;H(\cdot)), \quad \text{for } 0 \le t \le T, \quad (22)$$

where $c(P_t, t)$ represents the value of a standard European call option given by

$$c(P_t, t) = Se^{-\delta(T-t)}\aleph\left(d_1\left(P_t, T-t; K\right)\right) - Ke^{-r(T-t)}\aleph\left(d_2\left(P_t, T-t; K\right)\right)$$

and $\pi_c(S, t; H(\cdot))$ represents the early exercise premium defined by

$$\pi_c\left(P_t, t; H(\cdot)\right) = \int_t^T \left[\delta P_t e^{-\delta(\eta-t)}\aleph\left(d_1\left(P_t, \eta-t; H(\eta)\right)\right)\right.$$
$$\left. -rKe^{-r(\eta-t)}\aleph\left(d_2\left(P_t, \eta-t; H(\eta)\right)\right)\right] d\eta$$

with

$$d_1\left(P_t, s-t; \beta\right) = \frac{\ln\left(P_t/\beta\right) + \left(r-\delta+1/2\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}},$$
$$d_2\left(P_t, s-t; \beta\right) = d_1\left(P_t, s-t; \beta\right) - \sigma\sqrt{s-t}.$$

Proof of Proposition 6. The proof of the proposition can be derived along the one of Proposition (5). Here the payoff of the option to expand is

$$e^{-r(\tau-t)}\left(\alpha E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)} ds\right] - I_\alpha\right)^+.$$

That is, when the option is exercised at a random time τ, the manager will receive α quantity of the present value of the expected cumulative future cash flow at a cost of I_α. Of course the exercise should occur only if the difference between these two values is positive (positive gain for the firm).

For the analytical expression, we have

$$\begin{aligned}\overline{V}_H^{\exp}\left(P_t, t\right) &= \sup_{\tau\in\mathcal{S}_{t,T}} E_t\left[\int_t^\infty P_s e^{-r(s-t)} ds + e^{-r(\tau-t)}\left(\alpha E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)} ds\right] - I_\alpha\right)^+\right] \\ &= \frac{P_t}{r-\mu} + \frac{\alpha}{r-\mu} \times \sup_{\tau\in\mathcal{S}_{t,T}} E_t\left[e^{-r(\tau-t)}\left(P_\tau - \frac{1}{\alpha}(r-\mu) I_\alpha\right)^+\right]\end{aligned}$$

At the end we have to optimize the expectation

$$E_t\left[e^{-r(\tau-t)}\left(P_\tau - \frac{1}{\alpha}(r-\mu) I_\alpha\right)^+\right]$$

over τ in the time interval $[t, T]$ which corresponds the price of a standard American call option on P_t, with strike $\frac{1}{\alpha}(r-\mu) I_\alpha$ and maturity T. □

4.3. Combining abandonment and expansion decisions

Suppose now that the manager has the right to exercise one of the two options (option to abandon or to expand) within a finite time interval $[0, T]$ and consider a valuation date t within the finite time interval. There will be two unknown time-varying and interdependent EEBs. Each of the boundaries is linked to a specific side of the option, either the abandonment side or the expansion side. We denote by $\tilde{L}(\cdot)$ the abandonment option side EEB and $\tilde{H}(\cdot)$ the one of the expansion option side. The next proposition gives an expectation representation of the value of the firm.

Proposition 7. *Consider a firm with an option to expand activities by α, at an investment cost of I_α, or to abandon activities with a salvage value of B. The rational value of the firm , at time t, is given by*

$$\overline{V}^c_{\tilde{H},\tilde{L}}(P_t, t) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t \left[\int_t^\infty P_s e^{-r(s-t)} ds + e^{-r(\tau - t)} \left[\left(\alpha E_\tau \left[\int_\tau^\infty P_s e^{-r(s-\tau)} ds \right] - I_\alpha \right)^+ + \left(B - E_\tau \left[\int_\tau^\infty P_s e^{-r(s-\tau)} ds \right] \right)^+ \right] \right]. \quad (23)$$

The optimal exercise policy is given by

$$\tau_t = \inf \left\{ s \in [t, T] : P_s = \tilde{H}(s) \text{ or } P_s = \tilde{L}(s) \right\}.$$

The value $\overline{V}^c_{\tilde{H},\tilde{L}}(P_t, t)$ has the analytical expression as follows:

$$\overline{V}^c_{\tilde{H},\tilde{L}}(P_t, t) = \frac{P_t}{r-\mu} + \frac{1}{r-\mu} \left[\alpha C \left(P_t, t, (r-\mu) I_\alpha / \alpha, \tilde{H}(\cdot) \right) + P \left(P_t, t, (r-\mu) B, \tilde{L}(\cdot) \right) \right]. \quad (24)$$

The early exercise boundaries $\tilde{H}$ and $\tilde{L}$ solve the coupled recursive non-linear integral equations

$$\tilde{H}(t) - K_I^\alpha = C\left(\tilde{H}(t), t, K_I^\alpha, \tilde{H}(\cdot) \right) + \frac{1}{\alpha} P \left(\tilde{H}(t), t, K_B, \tilde{L}(\cdot) \right) \quad (25a)$$

$$K_B - \tilde{L}(t) = \alpha C \left(\tilde{L}(t), t, K_I^\alpha, \tilde{H}(\cdot) \right) + P \left(\tilde{L}(t), t, K_B, \tilde{L}(\cdot) \right) \quad (25b)$$

for all $t \in [0, T]$, subject to the boundary conditions $\lim_{t \uparrow T} \tilde{H}(t) = \max\{K_I^\alpha, (r/\delta) K_I^\alpha\}$

and $\lim_{t\uparrow T}\tilde{L}(t) = \min\{K_B, (r/\delta)K_B\}$. *At maturity* $\tilde{H}(T) = (r-\mu)I_\alpha/\alpha$ *and* $\tilde{L}(T) = (r-\mu)B$.

Remark 4.1. *Both optimal exercise boundaries* $\tilde{H}(t)$ *and* $\tilde{L}(t)$ *interact. This is emphasized by Equations (25a)-(25b). Consequently they may not be identical to those of two independent real options.*

Proof of Proposition 7. The expectation form of the value of the firm can be obtained using the same trick as in Proposition 5 and given the combined option's payoff

$$e^{-r(\tau-t)}\left[\left(\alpha E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)}ds\right]-I_\alpha\right)^+ + \left(B - E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)}ds\right]\right)^+\right].$$

The analytical expression of the value of the firm with the combined real options is derived by computing the expectation representation in Proposition (7). We have

$$\begin{aligned}
\overline{V}^{\mathrm{c}}_{\tilde{H},\tilde{L}}(P_t,t) &= \sup_{\tau\in\mathcal{S}_{t,T}} E_t\left[\int_t^\infty P_s e^{-r(s-t)}ds\right. \\
&\quad + e^{-r(\tau-t)}\left[\left(\alpha E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)}ds\right]-I_\alpha\right)^+\right. \\
&\quad \left.\left. + \left(B - E_\tau\left[\int_\tau^\infty P_s e^{-r(s-\tau)}ds\right]\right)^+\right]\right] \\
&= \frac{P_t}{r-\mu} + \sup_{\tau\in\mathcal{S}_{t,T}} E_t\left[e^{-r(\tau-t)}\left[\left(\frac{\alpha P_\tau}{r-\mu}-I_\alpha\right)^+ + \left(B-\frac{P_\tau}{r-\mu}\right)^+\right]\right] \\
&= \frac{P_t}{r-\mu} \\
&\quad + \frac{1}{r-\mu}\sup_{\tau\in\mathcal{S}_{t,T}} E_t\left[e^{-r(\tau-t)}\left[\alpha\left(P_\tau - \frac{1}{\alpha}(r-\mu)I_\alpha\right)^+ + ((r-\mu)B - P_\tau)^+\right]\right].
\end{aligned}$$

The problem is reduced to the optimization of

$$E_t\left[e^{-r(\tau-t)}\left[\alpha\left(P_\tau - \frac{1}{\alpha}(r-\mu)I_\alpha\right)^+ + ((r-\mu)B - P_\tau)^+\right]\right]$$

over τ in $[t,T]$.

At any exercise time τ, the corresponding payoff function is

$$e^{-r(\tau-t)}\left[\alpha\left(P_\tau - \frac{1}{\alpha}(r-\mu)I_\alpha\right)^+ + ((r-\mu)B - P_\tau)^+\right]$$

and is comparable the one of a Strangle contract developed in Chapter **??**. Thus similar materials can be used to derive the analytical formulas. □

5. Numerical results and analysis

This section presents numerical investigation of the valuation methodologies developed above. For illustration, consider a fictitious small firm called *LM Factory*. The firm has recently begun its activities and now is interested in determining its value. Assume that the firm has the opportunity to increase by 50% its current capacity at a cost of 150[11]. Abandoning the current activities can be considered when the profit is low, i.e., when the present value of cash flows falls below a threshold L. Once an abandonment is decided, the production machines and furniture can be sold for 140. The value of the cash flow is estimated to 20. The analysis of previous years of operation suggests a cash flow variance of 0.4 and a trend of 0.04. The risk-free interest rate is $r = 0.08$. At a first stage, we will consider that the manager has an infinite time horizon for these two decisions. At a second stage, the horizon will be considered finite and decisions can only occur within the next 5 years. The objective here is to examine the firm's behavior in different scenarios.

Starting with the values of the firm's opportunities under different scenarios, Figure 1 represents the values of the expansion, abandonment and combined real options as functions of the level of cash flow P_t. Of course, the single abandonment option is a decreasing function of P_t while the expansion option is an increasing function of the same variable. The curve of combined option is an upward-pointing curve, meaning that lower and higher values of P_t add value to the option. We can see that within the corridor $[L, H]$, the combined option is more valuable than its single counterparts.

[Insert Figure 1 about here]

Figure 2 shows the values of a single expansion option and combined options as functions of the decision threshold H. The single expansion option presents a hump by varying the decision threshold. It reaches its maximum at $H = 54.73$. This critical threshold is analytically given by ii) in Theorem 3.2. A hump also appears when plotting the curve of the combined option value as function of $\tilde{H}$. The maximum, here, is $\tilde{H} = 59.66$, meaning that the manager of a firm with both options exercises the expansion option later, i.e., at higher values of cash flows, compared to the one holding a single expansion option.

[Insert Figure 2 about here]

[11] All amounts are given in thousands of euros, unless otherwise specified. The choice of the parameters' values is made as to numerically highlight the behavior of the opportunities' values (real options).

Figure 3 is about the values of a single abandonment option and combined options as functions of the threshold L. The optimal decision level that maximizes the abandonment option can be graphically observed at $L = 2.45$ as given by ii) in Theorem 3.1. The maximum for the combined option is reached at a lower level $\tilde{L} = 2.08$. That is, when the cash flows decreases, the manager of the firm with the combined option may wait a little while longer before abandoning the activities compared to the manager with a single abandonment option. In contrast to Figure 2, a significant difference between the combined and single real option is observed. Analyzing Figure 1, we can see that the expansion side real option is very valuable for $P_t = 20$. Thus, the combined option may be significantly valuable in comparison with the single abandonment option.

[**Insert Figure 3 about here**]

Figure 4 presents a 3D graph of the value of the combined option as a multivariate function of both decision thresholds H and L. The surface of the function also presents a hump, reflecting the existence and uniqueness of the pair $(\tilde{L}, \tilde{H})$ that maximizes the value of the option. The maximum of the option is obtained for $\tilde{L} = 2.08$ and $\tilde{H} = 59.66$. These optimal thresholds are numerically determined because their analytical expressions are almost impossible to obtain.

[**Insert Figure 4 about here**]

Figure 5 shows the value of the option to abandon as a function of the salvage value of the firm. It is an increasing and unlimited function of the salvage value B. On the other side, the expansion option is a decreasing and limited function of the investment cost I_α as shown by Figure 6. In fact, higher investment costs can jeopardize the investment decision, thus lowers the value of the option.

[**Insert Figure 5 about here**]

[**Insert Figure 6 about here**]

We also compare decision thresholds of finite lived options with those of perpetual options. Figures 7 and 8 show how the early exercise boundaries behave when varying the time t for some chosen values of the maturity date (5 years, 10 years and infinity). As expected the decision thresholds are constant (for the perpetual option), decreasing with regard to the time t (for the finite-lived expansion options) and increasing with regard to the time t (for the finite-lived abandonment options). We can observe the convergence of the EEB to the constant decision threshold of the perpetual options when increasing the maturity date from 5 to 10 years, for example. This illustrates how a manager with a shorter finite lived option may exercise the real option far earlier than a manager with longer maturity. However the exercise policy, starting from $t = 0$, of the short-dated (5 years) maturity is identical to the one of the long-dated maturity (10 years) when starting from $t = 5$ (in the interval $[5, 10]$). Both graphs have the same shape.

[**Insert Figure 7 about here**]

[**Insert Figure 8 about here**]

Figures 9 and 10 show how EEBs of the combined option and those associated to single abandonment and expansion options differ from each other. The first figure represents the EEB of the upper side of the combined option with the EEB of an expansion option. The second one, consider those of the lower side of the combined option and the abandonment option. We can observe a difference between the EEBs suggesting that the value of combined options may be different from the sum two independent single real options (Abandonment + Expansion) . The manager holding the combined option may wait longer before exercising the option. This supports the conclusions in the case the perpetual options discussed earlier.

[**Insert Figure 9 about here**]

[**Insert Figure 10 about here**]

Table 1 gives the values of different real option contracts for different values of P_t. Recall that $\widetilde{V}$ is the value of the firm without any real option and generating, infinitely, cash flows driven by the process P. The notations v_L^{ab}, v_H^{exp} and $v_{\tilde{L},\tilde{H}}^{\text{abexp}}$ stand, respectively, for the value of the abandonment, expansion and combined options when the manager has infinite time horizon for the financial decisions. In the case of a finite time horizon[12] ($T = 5$ years), an overline is added and we have $\overline{v}_L^{\text{ab}}$, $\overline{v}_H^{\text{exp}}$ and $\overline{v}_{\tilde{L},\tilde{H}}^{\text{abexp}}$. The analysis of Table 1 indicates that infinite lived contracts are more valuable than their finite lived counterpart. The reason is that the additional leeway to delay the financial decisions increases the value of the option. Here also, the numerical values show that the combined option is not a simple sum of an abandonment and an expansion option.

[**Insert Table 1 about here**]

[12]The integral equation involved in the computation of the EEB is computed using subintervals of length $h = T/1000$. Using thinner subintervals would improve the approximation of the integral equations.

Table 2 shows how the values of the different opportunities (expansion and/or abandonment) vary when changing the maturity date T for $P = 10$, $r = 0.08$, $\delta = 0.04$, $\sigma = 0.4$, $\alpha = 0.5$, $I_\alpha = 150$ and $B = 140$. The values increase with the maturity date and converge to those of the corresponding opportunities with infinite time horizon (last line of the table). In fact, the time constraint shrinks the additional possibilities for the future cash flows scenarios. Thus, the managers are more likely to take the decisions earlier than they would do if they have more leeway in terms of time (see also Figures 7 and 8). Consequently, the value for waiting is lowered and the different opportunities lose value when maturity decreases.

[**Insert Table 2 about here**]

Table 3 analyzes the sensitivity of the value of the investment/disinvestment opportunities to the volatility (σ) underlying the future cash flows for $P = 10$, $r = 0.08$, $\delta = 0.04$, $T = 5$ (for finite lived opportunities), $\alpha = 0.5$, $I_\alpha = 150$ and $B = 140$. We can see, from the table, that the investment/disinvestment is highly sensitive to volatility in the future values of the cash flows. The value of the project significantly increases when σ increases. In other words, firms are more valuable under very uncertain market conditions. This highlight a significant advantage of the real option approach compared to the classical NPV approach. In this latter approach, it is often assumed that $\sigma = 0$.

[**Insert Table 3 about here**]

6. Conclusion

This paper extends the work of Dixit & Pindyck (1994) by considering the valuation of a firm holding simultaneously an option to abandon and an option to expand activities in the Black-Scholes's framework. The cases of finite and infinite-lived options were investigated. Expectation representations and analytical formulas for the firm's values were given. It is assumed that the firm can extend its activities when the generated cash flow reaches an upper decision level and abandon the activities when it drops below a lower level. The decision levels were firstly considered exogenous and, secondly, determined endogenously in order to optimize the investment/disinvestment decisions. These levels were shown to be constant for infinite-lived options and time varying for finite-lived options. Numerical results documented this behavior. It is also shown, numerically, that the coupled options is not a simple sum of two independent options. This indicates that a manager with coupled options may react differently from two independent managers, each of them holding a single option.

The methodology developed in this paper may be suitable for valuing firms in general and small firms in particular with significant opportunities to expand their activities (export). In fact, in the first years of activity, most firms' managers face situations where they consider a possible extension or abandonment of their activities. These decisions can be delayed infinitely in the future or, due to some constraints, to a maximum finite time horizon. The methodology developed can also account for valuing levered firms by appropriately adjusting the options' parameters.

Appendix A. First hitting time

In this appendix we derive the expressions of the probabilities of first hitting the upper barrier H and the lower barrier L respectively represented by $Q(\tau_H < \tau_L)$ and $Q(\tau_L < \tau_H)$. The expressions are used to determine the corresponding conditional expectations $E_t\left[e^{-r(\tau_H - t)} \mid \tau_H < \tau_L\right]$ and $E_t\left[e^{-r(\tau_L - t)} \mid \tau_L < \tau_H\right]$.

We first define the new process $(X_s)_{s \geq t \geq 0}$ by

$$X_s = \frac{1}{\sigma} \ln\left(\frac{P_s}{P_t}\right), \qquad P_t \text{ given.}$$

The process can be written as $X_s = \nu s + W_s$ with $\nu = \frac{1}{\sigma}\left(\mu - \sigma^2/2\right)$ and we have $P_s = P_t e^{\sigma X_s}$. The Girsanow theorem ensures that there exists a new probability measure for which the process X is a standard Brownian motion. Let h and l be, respectively, an upper and a lower barrier for X with $h = \frac{1}{\sigma}\ln(H/P_t)$ and $l = \frac{1}{\sigma}\ln(L/P_t)$. The inequality $l < 0 < h$ must hold at inception date t.

Let τ_h and τ_l be respectively the first hitting times of the barriers h and l by the process X and $\tau_{h,l} = \inf\{\tau_h, \tau_l\}$. Consequently

$$Q(\tau_H < \tau_L) \equiv Q(\tau_h < \tau_l) \quad \text{and} \quad Q(\tau_L < \tau_H) \equiv Q(\tau_l < \tau_h).$$

By means of the Laplace transform of the first hitting time $\tau_{h,l}$, it is well known[13] that

$$E_t\left[e^{-r(\tau_h - t)} \mid \tau_h < \tau_l\right] Q(\tau_h < \tau_l) = \frac{e^{\nu h} \sinh\left[\sqrt{\nu^2 + 2r}\,|l|\right]}{\sinh\left[\sqrt{\nu^2 + 2r}\,(h - l)\right]} \tag{26}$$

and

$$E_t\left[e^{-r(\tau_l - t)} \mid \tau_l < \tau_h\right] Q(\tau_l < \tau_h) = \frac{e^{\nu l} \sinh\left[\sqrt{\nu^2 + 2r}\,h\right]}{\sinh\left[\sqrt{\nu^2 + 2r}\,(h - l)\right]} \tag{27}$$

where

$$Q(\tau_h < \tau_l) = \frac{e^{-2\nu l} - 1}{e^{-2\nu l} - e^{-2\nu h}} \tag{28}$$

and

$$\begin{aligned} Q(\tau_l < \tau_h) &= 1 - Q(\tau_h < \tau_l) \\ &= \frac{1 - e^{-2\nu h}}{e^{-2\nu l} - e^{-2\nu h}}. \end{aligned} \tag{29}$$

[13] See Douady (1999)

In fact, $Q(\tau_h < \tau_l)$ (Equation (28)) is obtained using the fundamental theorem of martingales. We know that for any scalar ν, the process defined by

$$Y_s = e^{(-2\nu)(W_s - W_t) - \frac{1}{2}(-2\nu)^2(s-t)}$$

is a martingale such that

$$\begin{aligned} E_t\left[Y_{\tau_{h,l}}\right] &= e^{-2\nu l} + \left(e^{-2\nu h} - e^{-2\nu l}\right) Q(\tau_h < \tau_l) \\ &= 1. \end{aligned}$$

The expressions of $Q(\tau_h < \tau_l)$ and $Q(\tau_l < \tau_h)$ follow.

Recall that $h = \frac{1}{\sigma}\ln(H/P_t)$ and $l = \frac{1}{\sigma}\ln(L/P_t)$. Relying on Moraux (2009), we get more explicit form for Equations (26)-(29). For the first equation (Equation (26)) we have

$$\begin{aligned} E_t\left[e^{-r(\tau_h - t)} \mid \tau_h < \tau_l\right] Q(\tau_h < \tau_l) &= \frac{e^{\nu h}\sinh\left[\sqrt{\nu^2+2r}|l|\right]}{\sinh\left[\sqrt{\nu^2+2r}\,(h-l)\right]} \\ &= \left(\frac{P_t}{H}\right)^{-\frac{\nu}{\sigma}} \times \frac{\left(\frac{P_t}{L}\right)^{\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{P_t}{L}\right)^{-\frac{1}{\sigma}\sqrt{\nu^2+2r}}}{\left(\frac{H}{L}\right)^{\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{H}{L}\right)^{-\frac{1}{\sigma}\sqrt{\nu^2+2r}}} \\ &= \frac{\left(\frac{P_t}{L}\right)^{-\frac{\nu}{\sigma}}}{\left(\frac{H}{L}\right)^{-\frac{\nu}{\sigma}}} \times \frac{\left(\frac{P_t}{L}\right)^{\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{P_t}{L}\right)^{-\frac{1}{\sigma}\sqrt{\nu^2+2r}}}{\left(\frac{H}{L}\right)^{\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{H}{L}\right)^{-\frac{1}{\sigma}\sqrt{\nu^2+2r}}} \\ &= \frac{\left(\frac{P_t}{L}\right)^{-\frac{\nu}{\sigma}+\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{P_t}{L}\right)^{-\frac{\nu}{\sigma}-\frac{1}{\sigma}\sqrt{\nu^2+2r}}}{\left(\frac{H}{L}\right)^{-\frac{\nu}{\sigma}+\frac{1}{\sigma}\sqrt{\nu^2+2r}} - \left(\frac{H}{L}\right)^{-\frac{\nu}{\sigma}-\frac{1}{\sigma}\sqrt{\nu^2+2r}}} \\ &= \frac{\left(\frac{P_t}{L}\right)^{\theta_0} - \left(\frac{P_t}{L}\right)^{\theta_1}}{\left(\frac{H}{L}\right)^{\theta_0} - \left(\frac{H}{L}\right)^{\theta_1}} \end{aligned}$$

where

$$\begin{aligned} \theta_0 &= \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2} \\ &= -\frac{\nu}{\sigma} - \frac{1}{\sigma}\sqrt{\nu^2+2r} \end{aligned}$$

and

$$\theta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2}$$
$$= -\frac{\nu}{\sigma} + \frac{1}{\sigma}\sqrt{\nu^2 + 2r}.$$

The remain expressions are obtained using the same trick. For Equations (28) and (29), one can just remark that $\frac{-2\nu}{\sigma} = \theta_0 + \theta_1$.

Appendix B. Derivation of the analytical expression

To derive the analytical expression in Proposition 4, recall that:

$$V^c_{L,H}(P_t, t) = E_t[\Phi_{t,1} \mid \tau_H < \tau_L] \times Q(\tau_H < \tau_L) + E_t[\Phi_{t,2} \mid \tau_L < \tau_H] \times Q(\tau_L < \tau_H) \quad (30)$$

where

$$\Phi_{t,1} = \int_t^{\tau_L} P_s e^{-r(s-t)} ds + \alpha e^{-r(\tau_H - t)} E_{\tau_H}\left[\int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds\right] - I_\alpha e^{-r(\tau_H - t)}$$

and

$$\Phi_{t,2} = \int_t^{\tau_L} P_s e^{-r(s-t)} ds + B e^{-r(\tau_L - t)}.$$

Equation (30) can be evaluated by means of well-known Brownian motion properties. Let's start with the first expectation of the equation.

$$\begin{aligned}
E_t[\Phi_{t,1} \mid \tau_H < \tau_L] &= E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds \right.\\
&\quad \left. + \alpha e^{-r(\tau_H - t)} E_{\tau_H}\left[\int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds\right] - I_\alpha e^{-r(\tau_H - t)} \mid \tau_H < \tau_L\right]\\
&= E_t\left[\int_t^{\infty} P_s e^{-r(s-t)} ds \right.\\
&\quad \left. + e^{-r(\tau_H - t)}\left(\alpha E_{\tau_H}\left[\int_{\tau_H}^{\infty} P_s e^{-r(s-\tau_H)} ds\right] - I_\alpha\right) \mid \tau_H < \tau_L\right]\\
&= E_t\left[\frac{P_t}{r-\mu} + \left(\frac{\alpha H}{r-\mu} - I_\alpha\right) e^{-r(\tau_H - t)} \mid \tau_H < \tau_L\right]\\
&= \frac{P_t}{r-\mu} + \left(\frac{\alpha H}{r-\mu} - I_\alpha\right) \times E_t\left[e^{-r(\tau_H - t)} \mid \tau_H < \tau_L\right],
\end{aligned}$$

where the conditional expectation

$$E_t\left[e^{-r(\tau_H-t)} \mid \tau_H < \tau_L\right] = \frac{1}{Q\left(\tau_H < \tau_L\right)} \times \frac{\left(\frac{P_t}{L}\right)^{\theta_0} - \left(\frac{P_t}{L}\right)^{\theta_1}}{\left(\frac{H}{L}\right)^{\theta_0} - \left(\frac{H}{L}\right)^{\theta_1}}$$

is derived in Appendix A.

The second expectation of Equation (30) is obtained similarly.

$$\begin{aligned}
E_t\left[\Phi_2 \mid \tau_L < \tau_H\right] &= E_t\left[\int_t^{\tau_L} P_s e^{-r(s-t)} ds + B e^{-r(\tau_L-t)} \mid \tau_L < \tau_H\right] \\
&= E_t\left[\int_t^{\infty} P_s e^{-r(s-t)} ds \right. \\
&\quad \left. - e^{-r(\tau_L-t)} E_{\tau_L}\left[\int_{\tau_L}^{\infty} P_s e^{-r(s-\tau_L)} ds\right] + B e^{-r(\tau_L-t)} \mid \tau_L < \tau_H\right] \\
&= E_t\left[\frac{P_t}{r-\mu} + e^{-r(\tau_L-t)}\left(B - \frac{L}{r-\mu}\right) \mid \tau_L < \tau_H\right] \\
&= \frac{P_t}{r-\mu} + \left(B - \frac{L}{r-\mu}\right) E_t\left[e^{-r(\tau_L-t)} \mid \tau_L < \tau_H\right],
\end{aligned}$$

where

$$E_t\left[e^{-r(\tau_L-t)} \mid \tau_L < \tau_H\right] = \frac{1}{Q\left(\tau_L < \tau_H\right)} \times \frac{\left(\frac{P_t}{H}\right)^{\theta_0} - \left(\frac{P_t}{H}\right)^{\theta_1}}{\left(\frac{H}{L}\right)^{-\theta_0} - \left(\frac{H}{L}\right)^{-\theta_1}}. \tag{31}$$

See Appendix A for the derivation of (31).

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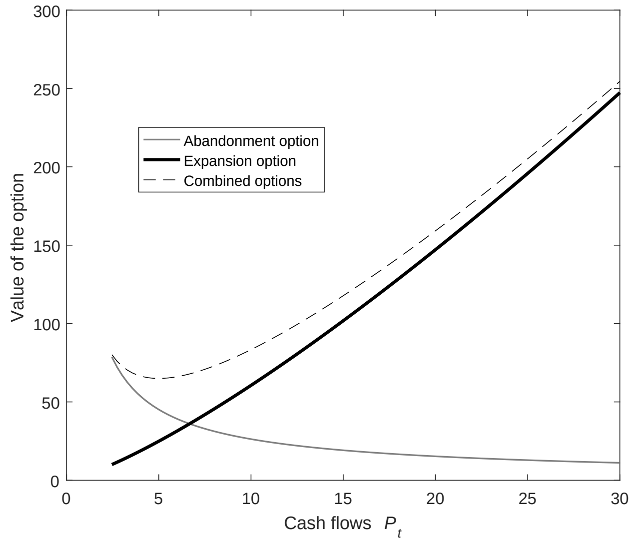

Fig. 1. Expansion, abandonment and combined options as functions of P_t

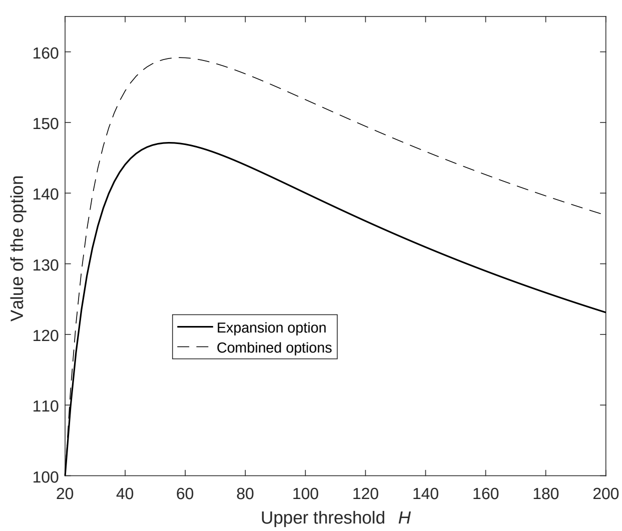

Fig. 2. Expansion and combined real options as functions of H.

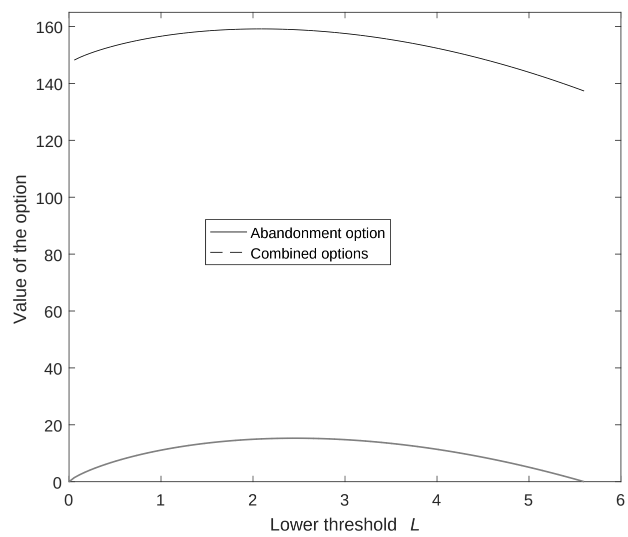

Fig. 3. Abandonment and combined real options as functions of L.

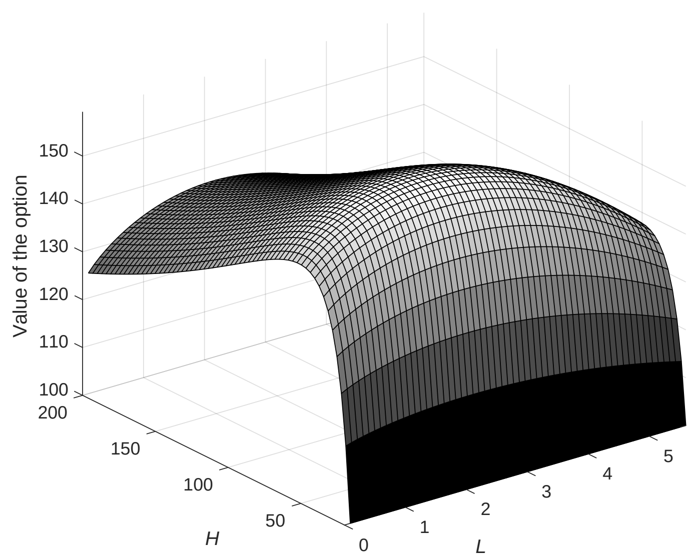

Fig. 4. Combined option as function of the thresholds L and H.

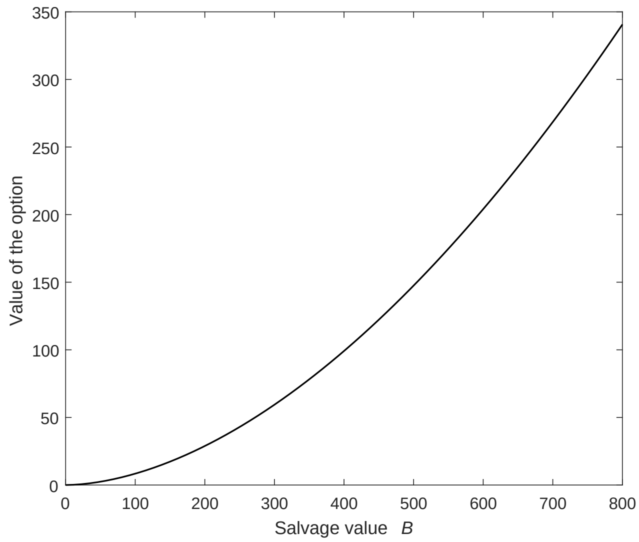

Fig. 5. Abandonment option as function of the salvage value B.

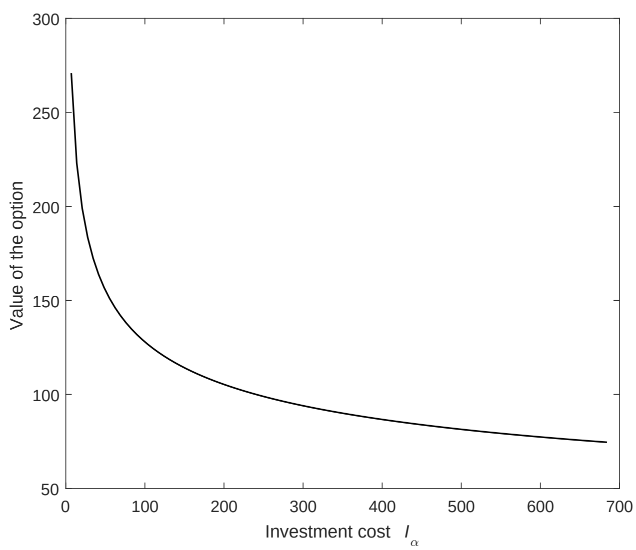

Fig. 6. Expansion option as function of the investment cost I_α.

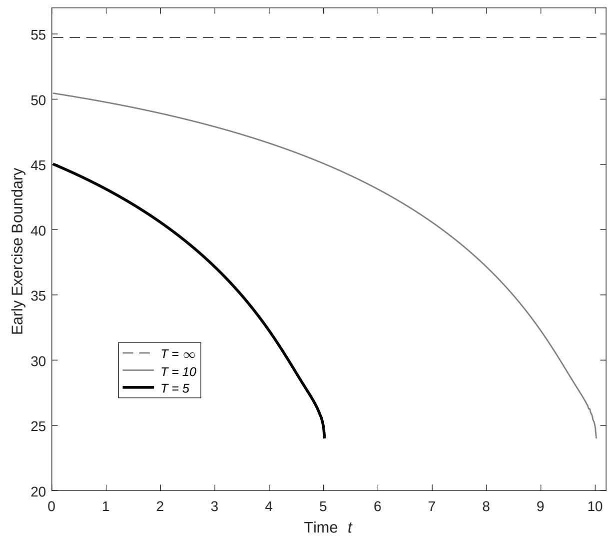

Fig. 7. Early Exercise Boundaries for different maturity dates.

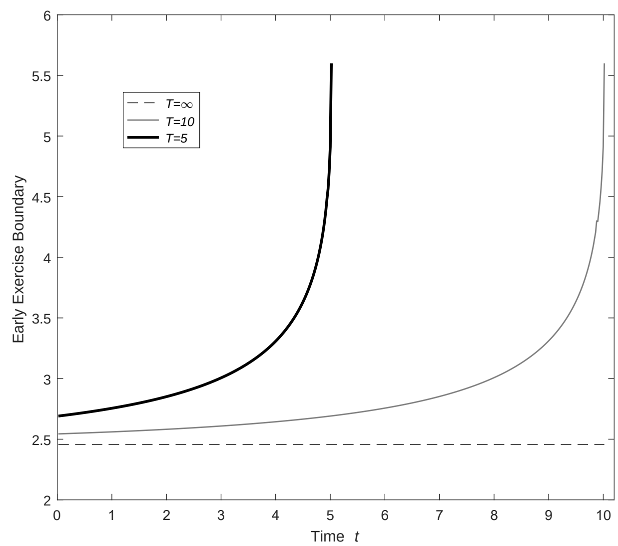

Fig. 8. Early Exercise Boundaries for different maturity dates.

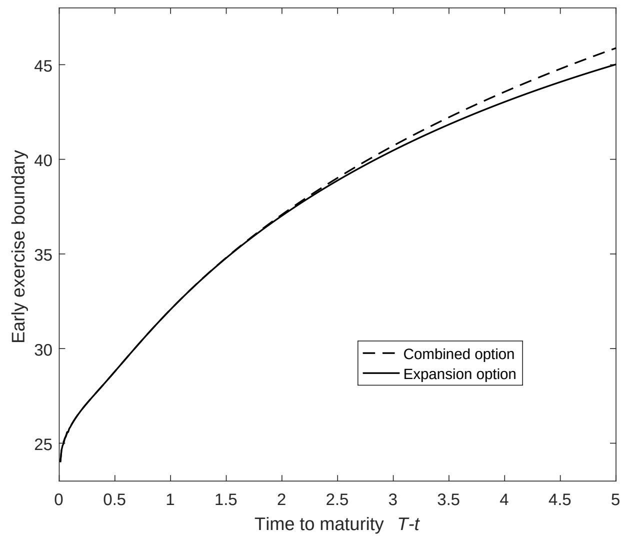

Fig. 9. EEBs of Combined option and Expansion option

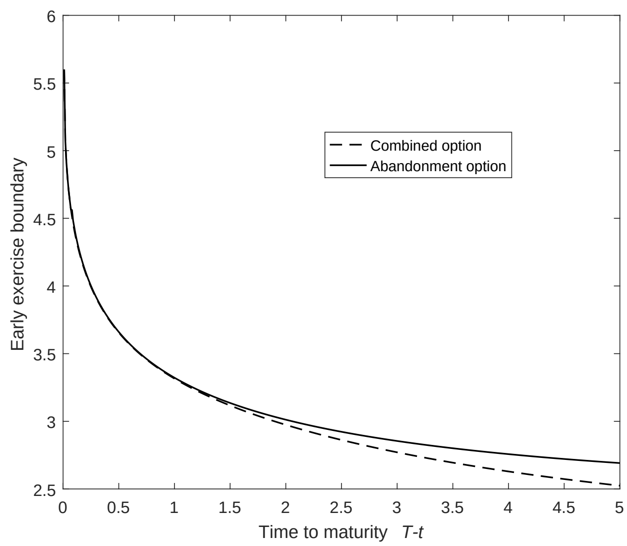

Fig. 10. EEBs of Combined option and Abandonment option

Table 1: Values of real options as function of the cash flow level P_t

P_t	$\widetilde{V}$	v_L^{ab}	$\overline{v}_L^{\text{ab}}$	v_H^{exp}	$\overline{v}_H^{\text{exp}}$	$v_{\tilde{L},\tilde{H}}^{\text{abexp}}$	$\overline{v}_{\tilde{L},\tilde{H}}^{\text{abexp}}$
3	75	67.23	65.65	12.96	2.00	72.90	66.69
5	125	45.12	38.03	24.92	7.96	64.97	45.72
10	250	26.26	14.29	60.55	36.87	83.36	51.14
15	375	19.14	6.83	101.78	77.92	117.76	84.74
20	500	15.29	3.71	147.13	125.75	159.17	129.44
25	625	12.84	2.20	195.80	177.92	205.12	180.06

Table 2: Values of real options as function of the maturity T

T	$\overline{v}_L^{\text{ab}}$	$\overline{v}_H^{\text{exp}}$	$\overline{v}_{\tilde{L},\tilde{H}}^{\text{abexp}}$
1	1.85	12.57	14.43
2	5.75	21.35	27.11
3	9.20	27.79	36.99
4	12.01	32.81	44.82
5	14.29	36.87	51.14
∞	26.26	60.55	83.36

Table 3: Values of real options as function of the volatility σ

σ	v_L^{ab}	$\overline{v}_L^{\mathrm{ab}}$	v_H^{exp}	$\overline{v}_H^{\mathrm{exp}}$	$v_{\tilde{L},\tilde{H}}^{\mathrm{abexp}}$	$\overline{v}_{\tilde{L},\tilde{H}}^{\mathrm{abexp}}$
10%	0.03	0.002	30.91	9.95	30.88	9.96
20%	3.83	1.18	40.69	18.92	43.70	20.11
40%	26.26	14.29	60.55	36.87	83.36	51.14