Bond sensitivities when the interest rates are near the zero lower bound

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Abstract

Zero Lower Bound (ZLB) and even negative values for the interest rates have been present in markets since the 2008 financial crisis. This situation has caused troubles on available well-established financial theory and tools (as common pricers) leading various authors to explore suitable term structures suitable for the ZLB setting.

Though having a bond price is highly desirable, from the perspective of position and risk managements, it is also of paramount importance to have the corresponding parameter sensitivities. As this aspect does not seem to be covered by the recent literature, then we aim here to contribute in this direction.

Therefore we first derive analytic approximations of the yield-rate level sensitivities, with respect to the shock affecting the underlying shadow rate. This finding is then used to provide high order sensitivities of any Zero-Coupon-Bond (ZCB) price. Our results may be applied to perform the hedging of bond portfolio by a portfolio linked to interest rates, as is very often required in practice.

Keywords: Bond sensitivities, Zero Lower Bound, shadow rate, Vasicek Model
JEL Classification: G12, G17.

Contents

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1 Introduction

1.1 Context and motivation

Zero Lower Bound (ZLB) and even negative values for the interest rates have been present in markets since the 2008 financial crisis. This situation has caused troubles on available well-established financial theory and tools (as common pricers), leading various authors [CR15], [FFLL15], [FLT16], [Kri15], [ML16], [MPRR15], [Wu13] to explore suitable term structures. Subsequently, interest rate models and related bond valuation suitable for the ZLB setting have emerged.

Though having a bond price is highly desirable, from the perspective of position and risk managements, it is also of paramount importance to have the corresponding parameter sensitivities. As this aspect does not seem to be covered by the recent literature, then we aim here to contribute in this direction with the intention to provide practical tools.

1.2 Literature and issue

In the classical, moderated and high interest rate regime levels, the bond sensitivities are first caught by the notion of (Fisher) duration and convexity, which underlies a restrictive assumption of parallel shift of the whole term structure. This is a more-and-less satisfactory approach if the intention is to manage risk in a conservative way. Duration and convexity, under the ZLB context, appear to be unsustainable when the interest rate levels are already near zero since a down parallel shift, with 10 bps for example, would lead to negative rates for too many maturities.

On the other hand, taking into account any arbitrary deformation of the curve is useful for the hedging perspective. The notion of stochastic duration and convexity has not appeared to provide a fully satisfactory solution to a bond sensitivity parameters and its related hedging [Wu00]. In contrast with an equity option, where the notion of delta and gamma are uniquely defined and well established, for a coupon-bearing-bond (linked to interest rates with various time-to-maturities) there are many possible ways to define the notion of sensitivity. It is common among practitioners to define the sensitivities of a given position (taken individually or at a portfolio level) as the resulting change value following changes of values of financial instruments or indexes contributing to the curve of interest rates used to value the considered position. From a theoretical consistency and perspective of portfolio management, it makes more sense to define the sensitivities directly with respect to the underlying risk(s) factor(s) governing the whole term structure.

As the deterministic duration and convexity have to be discarded under the ZLB framework, it remains for us to explore the deformation of the curve of interest rates based on one among the recent models proposed by the above mentioned authors. This is however challenging as each yield-rate is in general not analytically known and is at best approximately given by a highly nonlinear function of the underlying shadow rate.

1.3 Our Conceptual contribution

Our work is based on the ZLB interest rate modeling and bond pricing introduced by Krippner [Kri15], where the short rate is defined by means of an underlying state variable named as shadow rate. This last is assumed here to be governed by the celebrated one-factor Vasicek model for short rates, though the Krippner's approach remains to be valid for any Gaussian model.

First, we have derived analytic approximations of the sensitivities of any yield-rate (with a given maturity) with respect to the shock affecting the underlying shadow rate. Among motivations on such exploration is that under the ZLB model, the yield-rate appears to be a nonlinear function of the unobservable shadow rate, which consequently has to be filtered. Then these sensitivities allow to better grasp the error effect resulting from the yield-rate filtering. Moreover, they are also useful in the derivation of sensitivities related to prices or values of financial products linked to interest rates. It should be emphasized that actually we have obtained high order sensitivities, but not only the restricted first and second orders as for the well-established duration and convexity. As a striking fact here is that the sensitivities we introduce are devoted to represent the sensitivities of the yield rate at a given horizon fixed by the user herself. Both the usual delta/gamma for equity options and the classical duration/convexity for bonds do not incorporate the time-passage in their feature, as it would be if they are mainly intended for the hedging purpose.

Next, using these yield sensitivities, we are able to provide analytic approximations of high order sensitivities of any Zero-Coupon-Bond (ZCB) price with respect to the shock affecting the underlying shadow rate at a given time-horizon. We will put the emphasis on the fact that the ZCB price change during a given period of time has to be approximated by a one-variable polynomial function having these sensitivities as coefficients and the Gaussian shock affecting the shadow rate at the considered horizon as the underlying variable. It is also important to note that no assumption is used related to the size of this shock. This is in contrast with the common infinitesimal size implicitly assumed for the interest rate parallel shift related to the classical duration and convexity.

ZCBs are of paramount importance as they are involved in various financial instruments linked to interest rates, as Coupon-Bearing-Bond (CBB), Interest Rate Swap (IRS), Cap/Floor, Swaption .... High order sensitivities for the CBB prices are also provided here as a by-product of our result on yield-rate and ZCB.

Our interest on the derivation of high order sensitivities with respect to the underlying shadow rate relies on the willing to apply our finding in the hedging of a portfolio linked to interest rates by another portfolio also interest rates sensitive. This is indeed the need in practice, though in the literature very often the analyses are limited to the hedge of one individual position by another one single or sometime more instruments. Therefore the interplay between the change of the position to hedge and the change of the hedging instrument, in term of our introduced high order sensitivities, is considered and formulated in our work at least for the case of portfolios made by CBBs.

The key point in our derivation of the yield-rate sensitivities relies on a discretization of the integral form approximation as previously established in [Kri15]. Though the idea is simply on performing iterative derivations of the resulting suitable function, we are face with technical and lengthy calculations arising from the fact that this function is compounded by other highly nonlinear functions of the shadow rate. It should be noted that a direct derivation of ZCB sensitivities is more challenging than for the yield-rates.

1.4 Our empirical findings

In our numerical illustrations, we will show that the sensitivity value appears to be a decreasing quantity with respect to the order under consideration. Moreover, for the polynomial approximation of the yield-rate or zero-coupon price change, the shock power terms are counterbalanced by the factorial term in denominator, so preventing things to explode.

Even our development allows the shock to be of any arbitrary size, it is empirically clear that the quality of the change approximation (by the polynomial function involving sensitivities) depends on the order used and shock size. As expected, large shocks should require the need of high order sensitivities. Depending on the model parameters, we numerically show that shocks between -3 and 3 would be sufficient to consider when only relative changes of yield-rates between -20% and 20% are allowed to be happened at the time-horizon.

Though the hedger has a view of moderated shock sizes at the considered horizon, the requirement of high order sensitivities appears to be useful in connection of the complexity of the instrument at hand. Precisely it may be empirically observed that portfolio value change is better approximated by using high order sensitivities, while up to two order ones would be sufficient for a single ZCB.

Though our sensitivities are built with the intention to have applications on hedging under moderated shocks, it would be easy to extend our analyses in order to carry stressed situations characterized by large shocks.

1.5 Outline

Subsection 2.1 is first devoted to recall some known notions as the: shadow rate, Zero-coupon price, instantaneous forward rate and yield-rate. Also main facts related to the one-factor Vasicek (1-Vas) are recalled in Subsection 2.2. This is very useful as the 1-Vas is the main reference for our approach either as the underlying shadow rate or for comparison when the interest rate is in a high regime level. Next the integral expression of the yield-rate, based on the Krippner's forward approximation is presented in 2.3.

As the Krippner's approximation only leads to an integral expression of the yield rate, then with our first result in Theorem 1, we provide the discretization of this integral by using the Gauss-Legendre framework. Therefore from here one has an approximated closed formula allowing to get the yield-rate. This yield-rate appears to be a smooth function of the shock affecting the shadow rate at the considered time-valuation. It means that the computation of the sensitivities is mathematically reduced to perform the resulting derivatives. Though this is conceptually clear and simple, the detail computations appear to be challenging as actually one has to deal here with a highly nonlinear function of the underlying shock. Therefore we provide in our Theorem 2 the solution to the approximation of the yield-rate change based on the sensitivities introduced.

Once the sensitivities related to the yield-rate are obtained, then the next step which is carried in Subsection 3.4 is about the analogue quantities in the case of the zero-coupon bond price. A direct computations as performed in the case of yield rates seems not feasible. Therefore we have chosen in our main third result in Theorem 3 to derive the zero-coupon bond price sensitivities by exploring the sensitivities for the yield rates and using suitable truncations.

We apply our main results related to the zero-coupon bond prices in Subsections 3.5 and 3.6 to also derive sensitivities and change approximations for the coupon bearing bond and related portfolio.

Though our numerical experiments are commented in Section 4, the corresponding Tables and Plots are postponed in the Appendix part in Section 7.

Limit and further perspectives related to this work are presented in the conclusion part in Section 5.

2 Background

2.1 Shadow and instantaneous rates

According to Black [Bla95]

> *It is because currency is an option: when an instrument has a negative short rate, we can choose currency instead. Thus, we can treat the short rate itself as an option: we can choose a process that allows negative rates and can simply replace all the negative rates with zeros. We still have a process with a single number describing the state of the world: either the short rate (when it is positive or zero) or what the short rate would be without the currency option (when it is negative). We can call this number the "shadow short rate".*

Therefore it is now common to define the stochastic process

$$\Big(r_u(\cdot)\Big)_{u\geq 0}$$

associated with the instantaneous interest rate as

$$r_u(\cdot) \equiv \max\Big\{0; x_u(\cdot)\Big\} \tag{1}$$

where

$$\Big(x_u(\cdot)\Big)_{u\geq 0}$$

is an universal risk-driver which applies for both high-rates and low-rates environments. The instantaneous rate r_u in (1), is constrained to be positive in contrast with the shadow rate x_u which can take any real number value.

As followed in various [CR15], [CR16], [Kri13a], [MPRR15], the key point with (1) is that, instead of modeling the short rate evolution to be restricted to take only positive values, which in general leads to complex model, it seems more natural to directly model the shadow rate by any unrestricted process as the Ornstein Uhlenbeck one for example.

In this paper, we will focus on a shadow rate $\Big(x_t(\cdot)\Big)_{t\geq 0}$ governed by the famous one-factor Vasicek (1-Vas) model whose the dynamic is driven by the Stochastic Differential Equation (SDE)

$$dx_t(\cdot) = \kappa\big(\theta - x_t\big)dt + \sigma dW_t^{(\mathbb{Q})}(\cdot). \tag{2}$$

Here the real numbers κ, θ and σ represent respectively a mean reversion factor, a long run equilibrium and a volatility term. At least κ and σ are nonnegative constants. The dynamic in (2) is given under some risk neutral probability measure $\mathbb{Q}$ assumed to exist.
The dot notation as $y(\cdot)$ is used to emphasize on the uncertainty related to the quantity y. So $dx_t(\cdot)$ is suitable since, when being at the current time t, intuitively this quantity represents the difference $x_{t+\triangle}(\cdot) - x_t$, for some short time-step $\triangle$.

Though it is possible to consider more general and valuable models for the shadow rate, as the G2++ [BM06] and AFDNS's models [Ch-Di-Ru; 2011], we prefer in this paper to stick on an underlying rates driven by the famous Vasicek model (2). Among the reasons to deal with this benchmark model, is that it would allow to get easily a first insight of the complexity spanned by the nonlinearity arising with the use of a call-option transform as (1).

For t and τ, with

$$0 \leq t \qquad \text{and} \qquad 0 < \tau$$

let us denote by

$$\mathbf{P}(t, t+\tau)$$

the time-t price of the Zero-Coupon Bond (ZCB) having $T = t + \tau$ as a maturity. Said differently, $\mathbf{P}(t, t+\tau)$ represents the value of one currency unit paid at T as seen from t. By the fundamental theorem of asset pricing, one has

$$\mathbf{P}(t, t+\tau) \equiv \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^{t+\tau} r_u(\cdot)du\right)\Big|\mathcal{F}_t\right] \tag{3}$$

where the expectation is with respect to the risk-neutral probability measure $\mathbb{Q}$ and conditioned by the whole informations $\mathcal{F}_t$ available up to time-t.

Though the time-t price of the ZCB with the time-to-maturity τ is theoretically defined as in (3), for the practical point of view it is linked with the yield-rate $\mathbf{y}(t, t+\tau)$ according to

$$\mathbf{P}(t, t+\tau) = \exp\big[-\mathbf{y}(t, t+\tau)\tau\big] \tag{4}$$

or

$$\mathbf{y}(t, t+\tau) = -\frac{1}{\tau}\ln\big[\mathbf{P}(t, t+\tau)\big]. \tag{5}$$

Alternatively to dealing with the expectation operator as in (3), the ZCB price is also seen as closely linked to the instantaneous forward rate $f(t, t+u)$ according to

$$\mathbf{P}(t, t+\tau) = \exp\left(-\int_t^{t+\tau} \mathbf{f}(t, t+u)du\right) \tag{6}$$

such that

$$\mathbf{f}(t, t+u) = -\frac{\partial}{\partial \tau} \ln\big[\mathbf{P}(t, t+\tau)\big]\Big|_{\tau=u}. \tag{7}$$

In contrast with the yield-rates with various time-to-maturities which are very often quoted in market and consequently assumed to be observables, the instantaneous forward rates remain to be unobservables. However these last are introduced as useful objects to derive, at least theoretically, the yield rates. Indeed according to (5) and (6), the yield-rate may be obtained from the forward rate as

$$\mathbf{y}(t, t+\tau) = \frac{1}{\tau} \int_t^{t+\tau} \mathbf{f}(t, t+u) du. \tag{8}$$

In spite of the simplicity of the shadow rate dynamic (2), due to the non-linearity linked to the instantaneous short rate in (1), the computation of the zero-coupon price $\mathbf{P}(t, t+\tau)$ is challenging. We will come back on this aspect below.

2.2 Zero-coupon price under the 1-Vas model

In order to appreciate and compare our results with classical ones, we recall in this Subsection known results [Vas77] related to the zero-coupon price under the one-factor Vasicek (1-Vas) model.

Using the well-known Itô's lemma to the process in (2) one has

$$x_u(\cdot) = \exp\big[-\kappa(u-t)\big] x_t + \kappa\theta \mathbf{b}\big(u-t;\kappa\big) + \sigma \mathbf{b}^{\frac{1}{2}}\big(u-t; 2\kappa\big) Z_u(\cdot|t) \qquad \text{for } t < u \tag{9}$$

where

$$\mathbf{b}\big(u;\kappa\big) \equiv \Big(\frac{1}{\kappa}\Big)\Big(1 - \exp\big[-\kappa u\big]\Big) \tag{10}$$

and

$$Z_u(\cdot|t) \equiv Z_u(\cdot;\kappa|t) = \mathbf{b}^{-\frac{1}{2}}\big(u-t;2\kappa\big) \exp\big[-\kappa u\big] \int_t^u \exp\big[\kappa v\big] dW_v^{(\mathbb{Q})}(\cdot). \tag{11}$$

The random variable $Z_u(\cdot|t)$ is conditionally a standard Gaussian such that

$$\mathbb{E}_{\mathbb{Q}}\Big[Z_u(\cdot|t)\Big|\mathcal{F}_t\Big] = 0 \qquad \text{and} \qquad \mathbb{V}_{\mathbb{Q}}\Big[Z_u(\cdot|t)\Big|\mathcal{F}_t\Big] = 1.$$

Moreover the time-t Vasicek zero-coupon price

$$P^{(Vas)}(t, t+\tau) \equiv \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^{t+\tau} x_u(\cdot) du\Big)\Big|\mathcal{F}_t\Big],$$

based on the (shadow) rate x_u, is explicitly given by

$$P^{(Vas)}(t, t+\tau) = \exp\Big[-\mathbf{b}\big(\tau;\kappa\big)x_t - \frac{\sigma^2}{4\kappa}\mathbf{b}^2\big(\tau;\kappa\big) - \Big(\theta - \frac{\sigma^2}{2\kappa^2}\Big)\Big(\tau - \mathbf{b}\big(\tau;\kappa\big)\Big)\Big]. \tag{12}$$

A first main point for the derivation of (12) is that

$$\int_t^{t+\tau} x_u(\cdot) du = \mathbf{b}\big(\tau;\kappa\big)x_t + \theta\Big(\tau - \mathbf{b}\big(\tau;\kappa\big)\Big) + \sigma \int_t^{t+\tau} \mathbf{b}\big(t+\tau-u;\kappa\big) dW_u^{(\mathbb{Q})}(\cdot), \tag{13}$$

which results directly when integrating the expressions in (9). This last identity (13) shows that $\int_t^{t+\tau} x_u(\cdot) du$ is a Gaussian random random variable. Then the second point to get (12) is the well-known property

$$\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(X(\cdot)\Big)\Big] = \exp\Big(\mathbb{E}_{\mathbb{Q}}\big[X(\cdot)\big] + \frac{1}{2}\mathbb{V}_{\mathbb{Q}}\big[X(\cdot)\big]\Big)$$

which remains valid for any Gaussian random variable $X(\cdot)$.

Under the 1-Vas, it is well-established that the time-t forward rate with the time-to-maturity τ is given by

$$\mathbf{f}^{(Vas)}(t, t+u) = \exp[-\kappa u] x_t + \kappa\theta \mathbf{b}(u;\kappa) - \frac{1}{2}\sigma^2 \mathbf{b}^2(u;\kappa). \tag{14}$$

2.3 Zero-coupon price under the Krippner approach

Computation of the zero-coupon price $\mathbf{P}(t, t+\tau)$, as defined in (3), associated with an instantaneous rate r_t even driven by an underlying shadow rate x_t following the 1-Vas model as in (2) is challenging.

As mentioned by Krippner [Kri15] and according to the financial theory, the bound price is the solution to the PDE

$$\max\{0; x_t\} P(t, t+\tau) = \frac{\partial P}{\partial \tau}(t, t+\tau) + \kappa\big(\theta - x_t\big)\frac{\partial P}{\partial x_t}(t, t+\tau) + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x_t^2}(t, t+\tau) \tag{15}$$

with the boundary condition $P(t, t) = 0$. Several approximations can be found in the literature. Firstly, a Monte-Carlo and/or the control-variate approach has been used by Krippner [Kri13b]. Gorovoi and Linetsky [GL04] proposed an analytic expansion that involves the computation of special functions. Consequently, this solution may not be used on all numerical softwares. In [Kri13b] and [Kri15], Krippner has introduced an approximation of the forward rate which, according to (8), leads to the yield-rate and the ZCB values approximation. His approach provides a formula for the Zero Lower Bound instantaneous forward rate that applies to any Gaussian model but not only for the 1-Vas model.

Our derivation of the bond sensitivities relies on this mentioned result by Krippner (2013). As highlighted by Christensen and Rudebusch (2015), this Krippner (2013) framework should be viewed as not fully internally consistent and should be considered as simply an approximation to an arbitrage-free model. Of course, away from the ZLB with a negligible call option, the model will match the standard arbitrage-free term structure representation. It means that the tractability benefit with the Krippner's approach fully justified our willing to stick with this Krippner result which can be stated as follows.

Proposition 1 (Krippner)

Let us assume the shadow-rate to follow the 1-Vas model. Then the time-t yield rate for the time-to-maturity τ is (approximately) given by

$$\mathbf{y}(t, t+\tau) \approx y(t, t+\tau) \equiv \frac{1}{\tau}\int_0^\tau \mathbf{G}\big(u; x_t; \Theta\big) du, \qquad \tau > 0 \tag{16}$$

where

$$\mathbf{G}\big(u; x_t; \Theta\big) = \mathbf{F}\big(u; x_t; \Theta\big)\Phi\left(\frac{\mathbf{F}\big(u; x_t; \Theta\big)}{\mathcal{V}(u;\Theta)}\right) + \mathcal{V}(u;\Theta)\varphi\left(\frac{\mathbf{F}\big(u; x_t; \Theta\big)}{\mathcal{V}(u;\Theta)}\right) \tag{17}$$

$$\mathbf{F}\big(u; x_t; \Theta\big) = \exp\big[-\kappa u\big] x_t + \kappa\theta \mathbf{b}\big(u;\kappa\big) - \frac{1}{2}\sigma^2 \mathbf{b}^2\big(u;\kappa\big) \tag{18}$$

$$\mathcal{V}(u;\Theta) = \sigma \mathbf{b}^{\frac{1}{2}}\big(u; 2\kappa\big) \tag{19}$$

and

$$\Theta = (\kappa, \theta, \sigma) \tag{20}$$

Here Φ and φ are used to denote the CDF and PDF of the standard normal Gaussian random variable.

For convenience in the sequel we will always deal with the approximated yield rate $y(t, t+\tau)$ rather than with the (unattainable) exact value $\mathbf{y}(t, t+\tau)$.

The main key in the derivation (16) relies on the above mentioned Krippner's approximation for the instantaneous forward rate which, in the particular case of an underlying shadow rate x_t following the 1-Vas model, takes the form

$$\mathbf{f}(t, t+u) \approx \mathbf{f}^{(Vas)}(t, t+u) + \psi(t, t+u) \tag{21}$$

where

$$\begin{aligned}\psi(t, t+u) = &-\mathbf{f}^{(Vas)}(t, t+u) \\ &+\mathbf{f}^{(Vas)}(t, t+u)\Phi\left(\frac{\mathbf{f}^{(Vas)}(t, t+u)}{\sigma\mathbf{b}^{\frac{1}{2}}\left(u; 2\kappa\right)}\right) + \sigma\mathbf{b}^{\frac{1}{2}}\left(u; 2\kappa\right)\varphi\left(\frac{\mathbf{f}^{(Vas)}(t, t+u)}{\sigma\mathbf{b}^{\frac{1}{2}}\left(u; 2\kappa\right)}\right).\end{aligned} \tag{22}$$

When inserting this value of $\mathbf{f}(t, t+u)$ inside (8) then the approximation (16) arises immediately.

With (16), it appears that the time-t yield for the maturity τ is given by a term integral involving a highly nonlinear function such that there is no hope that it can be explicitly calculated. Therefore to derive numerical result, we propose here the use of the Legendre quadrature approach.

3 Main results

3.1 Closed form approximation for the yield-rate

Instead of considering a yield rate for a fixed time-to-maturity, in practical applications, there is the need to consider a whole yield-curve made by increasing maturities. In any case, the integrals (16) for various maturities leads to redundant part calculations so it is a good idea to try to optimize the approach by considering once a time main terms and then to derive the willing quantities from these previous terms.

Our first result is devoted to the derivation of the whole interest rate curve based on the discretization of the expression integral stated in (16).

Theorem 1 *Let τ be a nonnegative time-to-maturity such that*

$$0 \equiv \tau_0 < \tau_1 < \ldots < \tau_k < \ldots < \tau \equiv \tau_m$$

for some nonnegative integer m. Let us assume the shadow-rate to follow the 1-VM. Then the time-t yield for the maturity τ is approximately given by

$$\mathbf{y}(t, t+\tau) = \frac{1}{\tau}\sum_{k=1}^{m}\sum_{i=1}^{I}\theta_k\mathbf{G}\left(u_{i,k}; x_t; \Theta\right)w_i \tag{23}$$

where

$$\theta_k \equiv \frac{1}{2}\left(\tau_k - \tau_{k-1}\right) \tag{24}$$

and

$$u_{i,k} \equiv \theta_k\left(1 + a_i\right) + \tau_{k-1}. \tag{25}$$

The yield-rate expression (23) implicitly involves the use of the abscissas $\left(a_i\right)_{i\in\{1,\ldots,I\}}$ and weights $\left(w_i\right)_{i\in\{1,\ldots,I\}}$ associated to a setting of Legendre Gaussian quadrature integration assumed to be chosen in advance.

In practical implementation, when focusing on some standard maturities (as for examples, 3 months, 9 months, 1 years and so on) it may be assumed that $(\tau_k - \tau_{k-1}) \leq 1$.

Our result in Theorem 1 is written with the intention to compute the curve of interest rates

$$\mathbf{y}(t, t+\tau_1), \ldots, \mathbf{y}(t, t+\tau_m), \ldots, \mathbf{y}(t, t+\tau_M) \tag{26}$$

such that for each $m \in \{1, \ldots, M\}$ ones has

$$\mathbf{Y}(t, t+\tau_m) \equiv \mathbf{y}(t, t+\tau_m)\tau_m = \sum_{k=1}^{m} \theta_k \Big(\sum_{i=1}^{I} \mathbf{G}\big(u_{i,k}; x_t\big) w_i \Big) \tag{27}$$

To derive the yield curve in (26), the implementation may start by the computation of the $I \times M$-dimensional matrix made by the

$$\Big(u_{i,m}\Big)_{i \in \{1,\ldots,I\}; m \in \{1,\ldots,M\}} \tag{28}$$

which allows to get the matrix

$$\Big(\mathbf{G}\big(u_{i,m}; x_t\big)\Big)_{i \in \{1,\ldots,I\}; m \in \{1,\ldots,M\}}. \tag{29}$$

3.2 The yield-rate at a future horizon

Our purpose in this Subsection is to motivate to what extend we can focus on some levels of the Gaussian shocks in the sequel when considering the sensitivities.

As already mentioned in (9), under the 1-Vas model it is well-known that the shadow-rate at any horizon t, with $0 < t$, is given by

$$x_t(\cdot) = \exp\big[-\kappa t\big] x_0 + \kappa\theta \mathbf{b}\big(t; \kappa\big) + \sigma \mathbf{b}^{\frac{1}{2}}\big(t; 2\kappa\big)\varepsilon_t(\cdot) \tag{30}$$

where $\varepsilon_t(\cdot)$ is the standard Gaussian normal random variable. The shadow rate x_t is not an observed variable. Very often it is common to take as a proxy for it the available yield-rate with the shortest time-to-maturity. It implies that a view on the shortest yield-rate evolution may roughly considered as a view on the unobserved short rate. Then a translation in term of shock is possible. Indeed, when using 30, the shock at a future horizon t corresponding to the state relative realization

$$\rho_t(\cdot) = \frac{x_t(\cdot) - x_0}{x_0}$$

is given by

$$\varepsilon_t(\cdot) = \frac{1}{\sigma \mathbf{b}^{\frac{1}{2}}\big(t; 2\kappa\big)} \Big(\rho_t(\cdot) x_0 - \kappa \mathbf{b}\big(t; \kappa\big)(\theta - x_0) \Big). \tag{31}$$

To better grasp the meaning of this last relation and calibrate the domain of the shock ε let us consider four different cases for the 1-Vas model parameters

- **Case 1 :** US IR obtained from 1952 to 2004 ontained by using 3 months maturity such that

$$x_0 = 2.5\%, \quad \kappa = 18.171718\%, \quad \theta = 5.215587\%, \quad \sigma = 1.7599183\%$$

- **Case 2 :** EUR IR obtained from 1999 to 2007 obtained by using 3 months period daily Euro Interbank Offered Rate such that

$$x_0 = 2.5\%, \quad \kappa = 5.0692963\%, \quad \theta = 4.7513\%, \quad \sigma = 0.3891468\%$$

- **Case 3 :** US IR from Federal Resrve Bank at 31/12/2013 such that

$$x_0 = 2.5\%, \quad \kappa = 13.2103\%, \quad \theta = 6.8720\%, \quad \sigma = 2.3520\%$$

- **Case 4 :** EUR IR from ECB at 31/12/2013 such that

$$x_0 = 0.0001\%, \quad \kappa = 16.8991\%, \quad \theta = 4.1046\%, \quad \sigma = 1.2546\%$$

.

Note that the case 1 and case 2 correspond to the situation before the 2007 fincancial crisis and thus do not correspond to lower interest rates. The lasts ones correspond to low regime rates, in this paper we are working with the third case.

We are interested in the study of the function which associate the shock ε to the state relative variation given by

$$\varepsilon \mapsto \frac{r_t(.) - r_0}{r_0} = v$$

where $r_t(.)$ is the instantaneous interest rate which can be approximate by

$$r_t = \max\{0, x_t(.)\} \approx y(t, t + \tau^*)$$

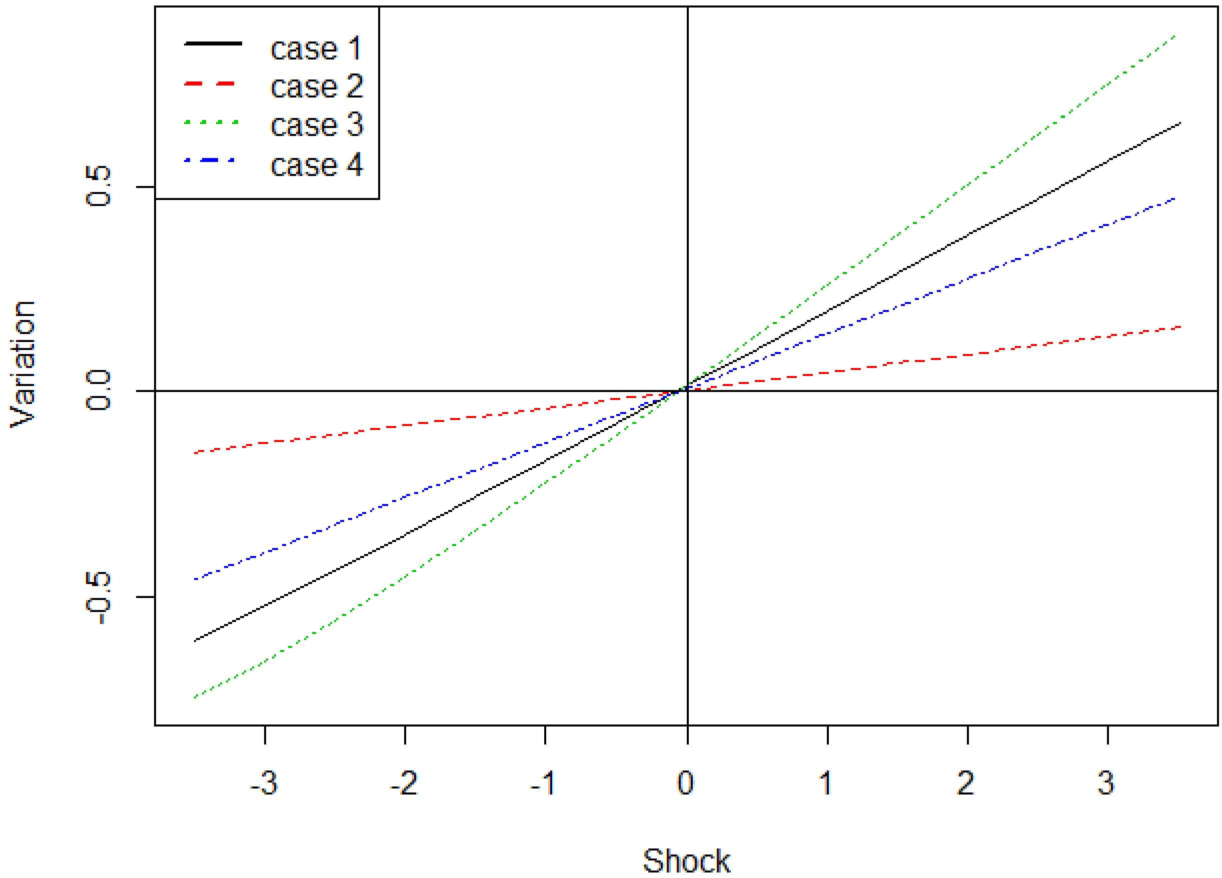
with $\tau^* = 6$ months.
Then, we can rewrite v such that

$$v = \frac{y(t, t + \tau^*) - y(t_0, t_0 + \tau^*)}{y(t_0, t_0 + \tau^*)}$$

with

$$y(t, t + \tau^*) = \frac{1}{\tau^*} \sum_{k=1}^{m} \sum_{i=1}^{I} \theta_k \mathbf{G}\big(u_{i,k}; \exp\big[-\kappa t\big] x_0 + \kappa\theta \mathbf{b}\big(t; \kappa\big) + \sigma \mathbf{b}^{\frac{1}{2}}\big(t; 2\kappa\big)\varepsilon_t(\cdot); \Theta\big) w_i$$

We can now plot the function $h : \varepsilon \mapsto v$ for the four different cases presented above, which give us the following graphic



First, one can notice the function h is always bijective(regardless of the paremeters choosen) which allows us to have the domain of the shock given the level of the state variation. We choose to consider a state variation v between -20% and 20% which corresponds to what observes a practitionner in practice. Since the function is bijective, when taking $v \in (-0.2, 0.2)$, for the third case, the corresponding shock ε is in $(-1.5, 1.5)$. If we consider the first case then the boundaries -0.2 and 0.2 for v are not reach for a shock in $(-3.5, 3.5)$. In the two other cases, the shock once again remains in $(-1.5, 1.5)$.

When considering yield-rates at a future horizon t, according to (16) and (30), it would be useful to consider the value of $\mathbf{G}\big(u; x_t(\cdot); \Theta\big)$ as depending on some shock $\varepsilon_t(\cdot)$. In other term, we have to analyze the mapping

$$\varepsilon_t(\cdot) \longmapsto \mathbf{G}\big(u; x_t(\cdot); \Theta\big).$$

3.3 Yield-rate sensitivities and its approximation change

In this section we assume to be at the present time 0 where the exact (or approximated) yield-curve

$$\Big(\mathbf{y}(0, 0 + \tau_m)\Big)_{m \in \{1,\dots,M\}}$$

is supposed to be available. Recall that by $\mathbf{y}(0, 0+\tau_m)$ we mean the time-0 yield rate with τ_m as a time-to-maturity. Let us consider a future horizon t, with

$$0 < t.$$

We are interested on the yield-curve values at this time horizon. Precisely our focus is on all changes

$$\mathbf{y}(t, t+\tau_m)(\cdot) - \mathbf{y}(0, 0+\tau_m). \tag{32}$$

The dot notation is used to recall that actually $\mathbf{y}(t, t+\tau_m)$ is a random quantity when viewed from the present time 0.

According to our result in Theorem 1 and especially the analysis in equation (23), the yield-rate $\mathbf{y}(t, t+\tau_m)(\cdot)$ appears to be an explicit function of the Gaussian shock $\varepsilon_t(\cdot)$ arising at the future time horizon. Actually this is a highly nonlinear function.

Our main achievement in this Subsection part is to build a one-variable polynomial approximation of the interest rate change, corresponding to any time-to-maturity τ_m. The underlying variable is named as shock. And the (constant) coefficients of such a polynomial function are refereed here to as the sensitivities of the considered yield-rate, with respect to a shock affecting the shadow rate. Consistently with the intuition and practical observation, these sensitivities highly depend on the time-horizon under consideration.

Under a view on the yield-rate with the shortest maturity, the polynomial approximation found below provides a quick look at the yield-curve at the future-time-horizon without performing a full calculation. But the main key application of our approximation is on the derivation of sensitivity prices for a zero-coupon. These last are useful either for a purpose of risk measurement or a position hedging.

In order to achieve this purpose lets see how $\mathbf{G}$ can be written as a polynomial function of the one variable ε. Recall that the expression of $\mathbf{G}(u_{i,m}, x_t, \Theta)$ is given by

$$\mathbf{G}\big(u_{i,m}; x_t; \Theta\big) = \mathbf{F}\big(u_{i,m}; x_t; \Theta\big)\Phi\left(\frac{\mathbf{F}\big(u_{i,m}; x_t; \Theta\big)}{\mathcal{V}(u_{i,m}; \Theta)}\right) + \mathcal{V}(u_{i,m}; \Theta)\varphi\left(\frac{\mathbf{F}\big(u_{i,m}; x_t; \Theta\big)}{\mathcal{V}(u_{i,m}; \Theta)}\right). \tag{33}$$

The $u_{i,m}$'s are given as in (25) of Theorem 1 in terms of the abscissas $\big(a_i\big)_{i\in\{1,\ldots,I\}}$ and weights $\big(w_i\big)_{i\in\{1,\ldots,I\}}$'s associated with the setting of Legendre Gaussian quadrature integration.

Now we set

$$\mathbf{F}\big(u_{i,m}; x_t; \Theta\big) = \alpha(u_{i,m})x_t + \beta(u_{i,m}) \quad \text{and} \quad \frac{\mathbf{F}\big(u_{i,m}; x_t; \Theta\big)}{\mathcal{V}(u_{i,m}; \Theta)} = \alpha^*(u_{i,m})x_t + \beta^*(u_{i,m})$$

where

$$\alpha(u_{i,m}) = \exp(-\kappa u_{i,m}), \quad \alpha^*(u_{i,m}) = \frac{\alpha(u_{i,m})}{\mathcal{V}(u_{i,m}; \Theta)},$$
$$\beta(u_{i,m}) = \kappa\theta\mathbf{b}(u_{i,m}, \kappa) - \frac{1}{2}\sigma^2\mathbf{b}^2(u_{i,m}, \kappa), \quad \beta^*(u_{i,m}) = \frac{\alpha(u_{i,m})}{\mathcal{V}(u_{i,m}; \Theta)}.$$

By applying the change of variable see in (30) and by momentarily ignoring some variable dependencies, then the expression in (33) can be simply written as

$$\begin{aligned}\mathbf{G}\big(u_{i,m}; x_t; \Theta\big) &= g\big(\varepsilon, \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big)\\ &= \big(\lambda_{i,m}\varepsilon + \nu_{i,m}\big)\Phi\big(\lambda^*_{i,m}\varepsilon + \nu^*_{i,m}\big) + v_{i,m}\varphi\big(\lambda^*_{i,m}\varepsilon + \nu^*_{i,m}\big)\end{aligned} \tag{34}$$

where

$$
\begin{aligned}
\lambda_{i,m} &= \Lambda_{i,m}(t, u_{i,m}, \Theta) \equiv \sigma \mathbf{b}^{\frac{1}{2}}(t, 2\kappa)\alpha(u_{i,m}),\\
\nu_{i,m} &= \Psi_{i,m}(t, u_{i,m}, \Theta) \equiv \alpha(u_{i,m})\Big(\exp^{-\kappa t} x_0 + \kappa\theta \mathbf{b}(t,\kappa)\Big) + \beta(u_{i,m}),\\
\lambda^*_{i,m} &= \Lambda^*_{i,m}(t, u_{i,m}, \Theta) \equiv \frac{\lambda_{i,m}}{\mathcal{V}(u_{i,m};\Theta)}, \quad \text{and} \quad \nu^*_{i,m} = \Psi^*_{i,m}(t, u_{i,m}, \Theta) \equiv \frac{\nu_{i,m}}{\mathcal{V}(u_{i,m};\Theta)}.
\end{aligned}
$$

For $i \in \{1, \ldots, I\}$ and $m \in \{1, \ldots, M\}$ we define the one-variable function

$$
\varepsilon \mapsto g_{i,m}(\varepsilon) = g(\varepsilon, \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}). \tag{35}
$$

In order to get our polynomial approximation we need to have the derivatives of g, up to an order N, and valued at $\varepsilon = 0$. These are defined by

$$
\begin{aligned}
g_{i,m}(0) &\equiv g\big(0; \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big)\\
&= \nu_{i,m}\Phi\big(\nu^*_{i,m}\big) + v_{i,m}\varphi\big(\nu^*_{i,m}\big)
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
g^{(1)}_{i,m}(0) &\equiv g^{(1)}\big(0; \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big)\\
&= \lambda_{i,m}\Phi\big(\nu^*_{i,m}\big) + \Big(\lambda^*_{i,m}\nu_{i,m} + a^*_{i,m}(0,1)v_{i,m}\Big)\varphi\big(\nu^*_{i,m}\big)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
g^{(2)}_{i,m}(0) &\equiv g^{(2)}\big(0; \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big)\\
&= \Big(b_{i,m}(0;2) + a^*_{i,m}(0;2)v_{i,m}\Big)\varphi(\nu^*_{i,m})
\end{aligned} \tag{38}
$$

and for $3 \leq n \leq N$

$$
\begin{aligned}
g^{(n)}_{i,m}(0) &\equiv g^{(n)}\big(0; \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big)\\
&= \Big(b_{i,m}(0;n) + a^*_{i,m}(0;n)v_{i,m}\Big)\varphi(\nu^*_{i,m}).
\end{aligned} \tag{39}
$$

where coefficients $a^*_{i,m}(k,n)$ and $b_{i,m}(k,n)$ are recursively obtained and displayed in the appendix.

We are now ready to state our second main result, related to the yield-rate and its approximation in term of related sensitivities.

Theorem 2 *Assume that at the time-horizon t the underlying short rate is impacted by some shock $\varepsilon_t(\cdot)$. Under the Krippner model* Kr-1-Vas, the yield-rate change, with the time-to-maturity τ_m, at this time-horizon t is given by the following N-th order level approximation

$$
\begin{aligned}
\mathbf{y}(t, t+\tau_m)\Big|_{\varepsilon_t(\cdot)} - \mathbf{y}(0, 0+\tau_m) \approx &\,\mathbf{sens\_yield}^{(0)}(0, t, \tau_m)\\
&+ \sum_{n=1}^{N} \frac{1}{n!}\mathbf{sens\_yield}^{(n)}(0, t, \tau_m)\varepsilon^n_t(\cdot)
\end{aligned} \tag{40}
$$

where N is a nonnegative integer and

$$\mathbf{sens\_yield}^{(0)}(0,t,\tau_m) \equiv \tau_m^{-1}\left(\sum_{i=1}^{I}\left[\sum_{k=1}^{m}\theta_k g_{i,k}(0)\right]w_i\right) - y(0,0+\tau_m) \quad (41)$$

and for $1 \leq n \leq N$,

$$\mathbf{sens\_yield}^{(n)}(0,t,\tau_m) \equiv \tau_m^{-1}\sum_{i=1}^{I}\left[\sum_{k=1}^{m}\theta_k g_{i,k}^{(n)}(0)\right]w_i. \quad (42)$$

It should be emphasized that by

$$\mathbf{sens\_yield}^{(n)}(0,t,\tau_m)$$

we mean the n-th order sensitivity of the yield-rate level with the time-to-maturity τ_m which prevails at the time-horizon t as viewed from the present time 0. This inclusion of the time-passage in the sensitivity is a main feature which distinguishes our present investigation from common sensitivities used and introduced the literature as the duration and convexity parameters

As can be seen in (40) we will content just to state an approximation of the yield-change without specifying more the size of the resulting error

$$\begin{aligned}\mathbf{error\_N}(t,\tau_m)(\cdot) &\equiv \mathbf{y}(t,t+\tau_m)(\cdot) - y(0,0+\tau) \\ &-\left(\mathbf{sens\_yield}^{(0)}(0,t,\tau_m) + \sum_{n=1}^{N}\frac{1}{n!}\mathbf{sens\_yield}^{(n)}(0,t,\tau_m)\varepsilon_t^n(\cdot)\right) \quad (43)\end{aligned}$$

though it is possible to do so, as a function of the size shock. The full explicit value is theoretically interesting but, as the resulting function is highly nonlinear and complex, from the practical point of view we think that it is enough to empirically analyze sample error approximation distributions corresponding to the shocks covering a range of plausible state variable relative changes. Nethertheless, our approximation is exact for the shock $\varepsilon = 0$

3.4 Sensitivities and approximation of the zero-coupon bond price

In many situations (as those which arise when dealing with coupon bearing bonds or interest swaps) one has to deal with the curve of Zero-Coupon Bond (ZCB) prices represented by

$$P(0,T_1),\ldots,P(0,T_m),\ldots,P(0,T_M)$$

for some nonnegative integer M and where the maturities are supposed to be increasing in the sense that

$$0 < T_1 < \ldots < T_m < \ldots < T_M.$$

Recall that each $P(0,T_m)$ is defined as

$$P(0,T_m) = \exp\Big[-y(0,T_m)T_m\Big].$$

Let us consider a future horizon t before the next first coupon, that is

$$0 < t < T_1$$

At the time-horizon t we are face with dealing the curve

$$P(t,T_1)(\cdot),\ldots,P(t,T_m)(\cdot),\ldots,P(t,T_M)(\cdot)$$

which is of course unknown when viewed from the present time 0. Actually the issue is on analyzing each m-th zero-coupon price change

$$P(t,T_m)(\cdot)-P(0,T_m)$$

which, by setting

$$\tau_m \equiv T_m - t$$

is nothing else than

$$P(t,t+\tau_m)(\cdot)-P(0,t+\tau_m). \tag{44}$$

As the zero-coupon price $P(t,t+\tau_m)(\cdot)$ is associated with the yield-rate $y(t,t+\tau_m)(\cdot)$ then it appears that we may benefit from our analysis performed in (40). In any case, the ZCB price change (44) depends on the shock $\varepsilon_t(\cdot)$ which arises at the future-horizon t. This dependence is given by a highly nonlinear function and consequently, as in the case of the yield-rate, we are interested on deriving a one-variable polynomial approximation. The variable is the underlying shock and the coefficients of such polynomial function are referred to as the ZCB price sensitivities. By sensitivity here, we mean the price change result with respect to one-unit shock affecting the underlying shadow rate. The point here is that the ZCB price sensitivities are defined by means of the yield-rate sensitivities introduced in the previous Subsection 3.1.

To define our ZCB price sensitivities let us set the following short notations:

$$\begin{aligned}\mathcal{Y}_0 &\equiv \mathbf{sens\_yield}^{(0)}(0,t,\tau_m)+y(0,0+\tau_m),\\ \mathcal{Y}_n &\equiv \mathbf{sens\_yield}^{(n)}(0,t,\tau_m) \qquad \text{for } 1\le n\le N.\end{aligned}$$

Then the price sensitivities of the ZCB, with the maturity $t+\tau_m$, at the future-time-horizon t and as viewed from the present time 0 are defined by

$$\mathbf{sens\_ZC}^{(0)}(0,t,\tau_m) \equiv \exp\big[-\mathcal{Y}_0\tau_m\big]-P(0,t+\tau_m) \tag{45}$$

$$\mathbf{sens\_ZC}^{(n)}(0,t,\tau_m) \equiv \Big\{Q_n(0)\Big\}\exp\big[-\mathcal{Y}_0\tau_m\big], \quad \text{for } 1\le n\le N. \tag{46}$$

The identity (46) makes use of the valuation at 0 of the N-th order polynomial function

$$Q_n(\varepsilon)=\sum_{k=0}^{N}\gamma_{k,n}\varepsilon^k, \quad \text{for } 1\le n\le N. \tag{47}$$

The integer N, assumed to be sufficiently large, arises from the technical consideration that actually we derive the ZCB price sensitivities by making use of level sensitivities for the yield-rates. The coefficients $(\gamma_{k,n})_n$'s involved in the polynomial function Q_n is recursively defined as follows

$$\begin{aligned}\gamma_{k,1} &= (k+1)a_{k+1} \quad 0\le k\le N-1, \quad a_k=-\frac{1}{k!}\mathcal{Y}_k\tau_m\\ \gamma_{N,1} &= 0\end{aligned}$$

and for $2\le n\le N$ one has

$$\begin{aligned}\gamma_{k,n} &= (k+1)\gamma_{k+1,n-1}+\beta_{k,n} \quad \text{for} \quad 0\le k\le N-1\\ \gamma_{N,n} &= \beta_{N,n}\end{aligned}$$

where

$$\beta_{k,n} = \sum_{j=1}^{k} \gamma_{j,1}\gamma_{k-j,n-1}.$$

The expected approximation for the ZCB price change (44), which is our third main result, can be now stated.

Theorem 3 *Assume that at the time-horizon t the underlying short rate is impacted by some shock $\varepsilon_t(\cdot)$. Under the* Kr-1-Vas *model, the price change, of the ZCB with the maturity T_m, may be approximated by the following N-th order polynomial function*

$$P(t, t+\tau_m)\Big|_{\varepsilon_t(\cdot)} - P(0, t+\tau_m) \approx \mathbf{sens\_ZC}^{(0)}(0, t, \tau_m) + \sum_{n=1}^{N} \frac{1}{n!} \mathbf{sens\_ZC}^{(n)}(0, t, \tau_m)\varepsilon_t^n(\cdot) \quad (48)$$

where n is and nonnegative integer not too large in the sense that $1 \leq n \leq N$. Here N is a nonnegative integer for which the remainder order N approximation for the yield rate is considered as small enough.

From the implementation perspective, either for the yield-rates curve or for the zero-coupon prices curve, we have to consider the matrices

$$\Big(\mathbf{sens\_yield}^{(n)}(0, t, \tau_m)\Big)_{m\in\{1,\ldots,M\};\ n\in\{0,\ldots,N\}}$$

and

$$\Big(\mathbf{sens\_ZC}^{(n)}(0, t, \tau_m)\Big)_{m\in\{1,\ldots,M\};\ n\in\{0,\ldots,N\}}.$$

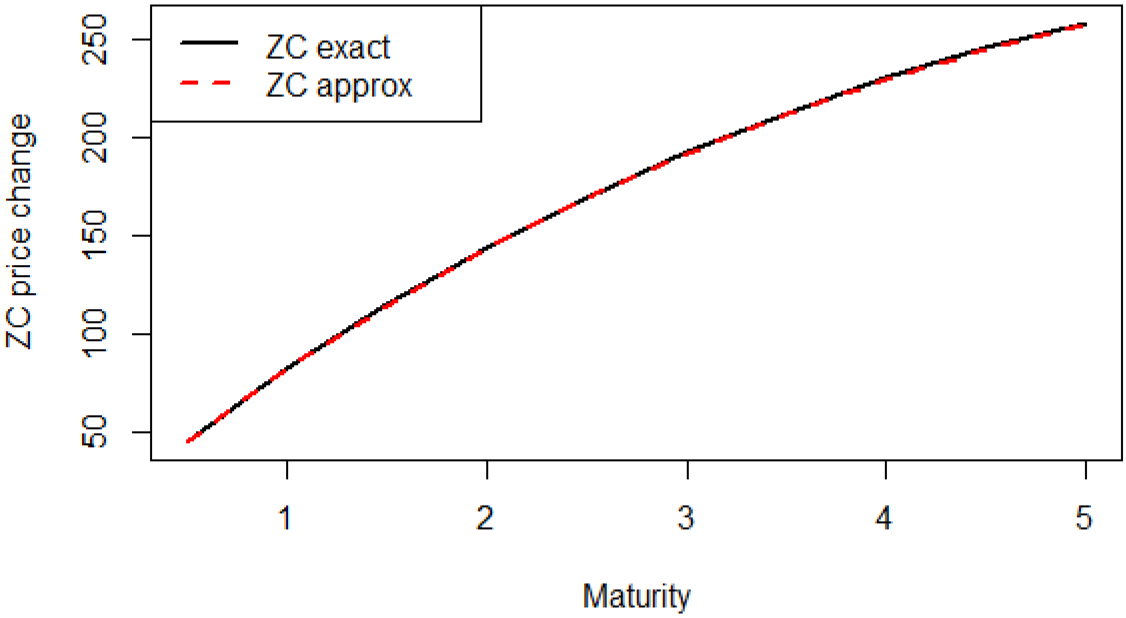
To have an idea of how our approximation is close enough to the real zero-coupon price change, we plot, for some shocks in $(-2, 2)$, the zero-coupon price curve obtained with our approximation and the one with the exact zero-coupon price. Various numerical illustrations both for the ZCB sensitivities and the related approximation are presented in section (7).

For the followings graphs, the 1-Vas parameters are

$$x_0 = 2.5\%, \quad \kappa = 18.171718\%, \quad \theta = 5.215587\%, \quad \sigma = 1.7599183\%$$

. The ZC price change are given in basis point, the black line correspond to the real ZC price change and the dot red line is the ZC price change obtained with our approximation.

Figure 1: $\varepsilon = -1.5$



When the shock is negative, here $\varepsilon = -1.5$, this result in a gain, which grows with the maturity.

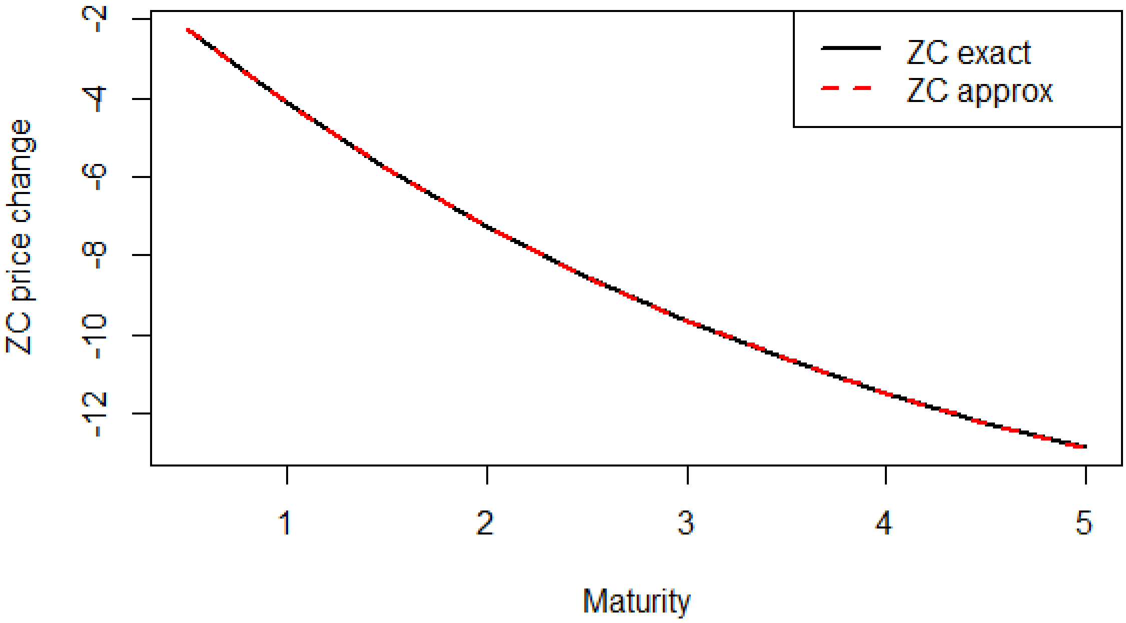
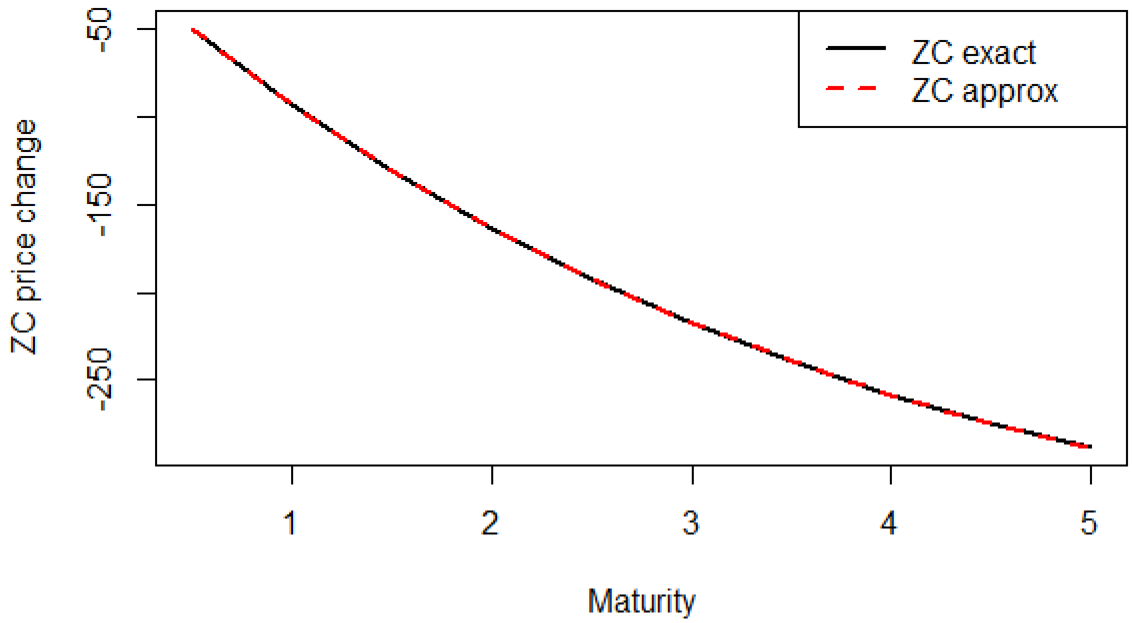


Figure 2: $\varepsilon = 0$

When there is no shock, the owner of the ZCB lost money on his position due to time passing.

Figure 3: $\varepsilon = 1.5$



A positive shock leads to a loss on the position which grows with the maturity.

The gain and the loss on a position are symetric for a symetric shock for a fixed maturity, in either case, the lost, respectively the gain, increase when the maturity grows. The situation where there is no shock also leads to a loss due the time passage.

3.5 Sensitivities and approximation of a Coupon-Bearing-Bond price

A Coupon-Bearing-Bond (CBB) is a debt security such that the issuer owes to the holders a debt and, depending on the terms of the considered bond, is obliged to pay interest (often named a coupon) and/or repay the principal at a later date, termed as maturity. In this work we consider vanilla bonds and assume that the issuer cannot default until the maturity. A CBB can be viewed as a series of ZCBs, each with a different maturity date. Among characteritics of a given CBB is its face value (usually 100 or 1000) and the coupon-rate, as for example 0.1% which is low under the ZLB environment.

For convenience we set

$$f \equiv \text{face value} \quad \text{and} \quad c \equiv \text{coupon-rate}.$$

and, as in the previous section, consider increasing times as

$$0 < t < T_1 < \cdots < T_m < \cdots < T_M. \tag{49}$$

Here t is a future time-horizon before the payment date T_1 of the first coupon. The T_m's should be seen as the CBB times coupon payment, such that T_M is the maturity.

Usually, the time-t price of the considered CBB price is given by

$$\begin{aligned}\mathbf{Price\_CBB}_t &= \sum_{m=1}^{M} C_m \times P(t, T_m) \\ &= \sum_{m=1}^{M} C_m \times P(t, t+\tau_m) \qquad \text{for } \tau_m = T_m - t \end{aligned} \tag{50}$$

where

$$\begin{aligned} C_1 &= f \times c \times (T_1 - t) \\ C_m &= f \times c \times (T_m - T_{m-1}) \quad \text{ for } 2 \le m \le (M-1) \\ C_M &= f \times \Big(1 + c \times (T_M - T_{M-1})\Big). \end{aligned}$$

The CBB price change under the Kr-1-Vas depends on a nonlinear fashion on the shock $\varepsilon_t(\cdot)$ arising at the future horizon t. As a linear combination of ZCBs prices, the CBB price is a highly nonlinear function of the shock. The approximation performed in the previous subsection 3.4 for the ZCB price change allows us to approximate the CBB price change. This is done by introducing the CBB sensitivities according to

$$\mathbf{sens\_CBB}^{(p)}\big(0, t, \mathcal{T}_M\big) = \sum_{m=1}^{M} C_m \times \mathbf{sens\_ZC}^{(p)}(0, t, \tau_m) \tag{51}$$

with $\mathcal{T}_M \equiv \big(\tau_1, \ldots, \tau_m, \ldots, \tau_M; f, c\big)$ where the $\mathbf{sens\_ZC}^{(n)}(0, t, \tau_m)$'s are defined from (45) and (46). Identity (51) holds for all positive integers n, and means that the CBB sensitivities are given in term of sensitivities for the ZCBs.

Now the result for the CBB change and its price approximation may be stated.

Proposition 2 *Assume that at the time-horizon t the underlying short rate is impacted by some shock $\varepsilon_t(\cdot)$. Under the Kr-1-Vas, the change of the CBB price is given by the following approximation*

$$\mathbf{Price\_CBB}_t(\cdot) - \mathbf{Price\_CBB}_0 \approx \sum_{p=0}^{n} \frac{1}{p!} \mathbf{sens\_CBB}^{(p)}(0, t) \varepsilon_t^p(\cdot). \tag{52}$$

To derive the CBB price sensitivities we have used the sensitivities for the ZCB prices, which themselves have been obtained from the yields sensitivities. Therefore the CBB sensitivities appear to be an approximation derived from various approximation, necessarily leading to a lost of precision. We think that increasing the order of the sensitivities contributes to improve the sensitivities essentially if one has a hedging strategy as a target.

3.6 Portfolio of Coupon-Bearing-Bonds

Let us denote by $\mathcal{B}_t$ the present time-t value of a portfolio made by CBBs in long or short positions. We assume there are I types of CBBs B_i in long position and $I*$ types of CBBs B^*_{i*} in short position inside our portfolio.

In the following, we considered a portfolio made of n_i CBBs of type i each worth $B_{t,i}$ and n_{i^*} CBBs of type i^* worth B^*_{t,i^*}. Then the time-t value of this portfolio can be written as

$$\mathcal{B}_t = \sum_{i=1}^{I} n_i B_{t,i} - \sum_{i^*=1}^{I^*} n_{i^*} B^*_{t,i^*} \tag{53}$$

Coupons of the CBB $B_{,i}$, with the maturity T_{M_i}, face value f_i and a coupon rate c_i are paid at increasing times

$$\mathcal{T}(i) = (T_1(i), \ldots, T_{M_i}(i)).$$

Similarly coupons of the CBB $B^*_{,i^*}$ with the maturity $T_{M_{i^*}}$, face value f_{i^*} and a coupon rate c_{i^*} are paid at increasing times

$$\mathcal{T}^*(i^*) = (T^*_1(i^*), \ldots, T^*_{M_{i^*}}(i^*))$$

In this paper we consider that no coupons has been paid before the time-horizon t at which the portfolio manager has a view about a possible market movement. So it is assumed that

$$0 < t < \min\{T_1(i), \ldots, T_{M_i}(i), T^*_1(i^*), \ldots, T^*_{M_{i^*}}(i^*)\}$$

Here we want to analyze the portfolio change value

$$\begin{aligned}\textbf{change\_port\_CBB}_{0,t}(\cdot) &= \mathcal{B}_t(\cdot) - \mathcal{B}_0 \\ &= \sum_{i=1}^{I} n_i \Big(B_{t,i}(\cdot) - B_{0,i}\Big) - \sum_{i^*=1}^{I^*} n_{i^*} \Big(B^*_{t,i}(\cdot) - B^*_{0,i^*}\Big)\end{aligned} \tag{54}$$

Since the portfolio CBB change is a linear combination of the CBBs change $B_{t,i}(\cdot) - B_{0,i}$ and $B^*_{t,i}(\cdot) - B^*_{0,i}$, then the sensitivities related to the portfolio would result from the expression in (51).

To get the approximation of the portfolio change value, there is the need to introduce the following short notations:

$$\begin{aligned}\textbf{sens\_CBB}_\mathbf{i}{}^{(n)}(0,t,\mathcal{T}_i) \qquad &\text{with} \qquad \mathcal{T}_i = \big(\mathcal{T}(i), f_i, c_i\big) \\ = \quad & f_i c_i \times \textbf{sens\_ZC}^{(n)}(0,t,\tau_1(i)) \\ + & \sum_{j=2}^{M_i-1} f_i c_i \times \textbf{sens\_ZC}^{(n)}(0,t,\tau_j(i)) \\ + & f_i * \Big(1 + c_i * \tau_{M_i}(i)\Big) \times \textbf{sens\_ZC}^{(n)}(0,t,\tau_{M_i}(i))\end{aligned}$$

$$\begin{aligned}\textbf{sens\_CBB}^*_{\mathbf{i^*}}{}^{(n)}(0,t,\mathcal{T}^*_{i^*}) \qquad &\text{with} \qquad \mathcal{T}^*_{i^*} = \big(\mathcal{T}^*(i^*), f^*_{i^*}, c^*_{i^*}\big) \\ = \quad & f_{i^*} c_{i^*} \times \textbf{sens\_ZC}^{(n)}(0,t,\tau^*_1(i^*)) \\ + & \sum_{j=2}^{M_{i^*}-1} f_{i^*} c_{i^*} \times \textbf{sens\_ZC}^{(n)}(0,t,\tau^*_j(i^*)) \\ + & f_{i^*} \Big(1 + c_{i^*} \tau^*_{M_{i^*}}(i)\Big) \times \textbf{sens\_ZC}^{(n)}(0,t,\tau^*_{M_{i^*}}(i^*))\end{aligned}$$

$$\textbf{Change\_B}_\mathbf{i}\big(n, \varepsilon(\cdot)\big) \equiv \sum_{p=1}^{n} \frac{1}{p!} \textbf{sens\_CBB}_\mathbf{i}{}^{(p)}(0,t) \varepsilon^p_t(\cdot)$$

$$\mathbf{Change\_B^*_{i^*}}\big(n,\varepsilon(\cdot)\big) \equiv \sum_{p=1}^{n} \frac{1}{p!}\mathbf{sens\_CBB^{*\,(p)}_{i^*}}(0,t)\varepsilon_t^p(\cdot).$$

Using all of these notations, we can now state the following result.

Proposition 3 *Assume that at the time-horizon t the underlying short rate is impacted by some shock $\varepsilon_t(\cdot)$. Under the* Kr-1-Vas *model, at the order n, the portfolio change value may be approximated as*

$$\mathbf{change\_port\_CBB}_{0,t}(\cdot) \approx \sum_{i=1}^{I} n_i \times \mathbf{Change\_B_i}\big(n,\varepsilon(\cdot)\big) - \sum_{i^*=1}^{I^*} n_{i^*} \times \mathbf{Change\_B^*_{i^*}}\big(n,\varepsilon(\cdot)\big). \quad (55)$$

4 Numerical experiments

4.1 Zero-coupon price

In the sequel for the considered calibration situation, the illustrations are just limited to a short horizon $t = 10$ days. We are interested to derive numerical values of errors approximations. For all tables (1) to (9), in the first column is given some possible values of the shock ε that belong in $(-2,2)$. The exact zero-coupon price change, withrespect to the given shock is displayed in the second column in basis point. It is obtained by computing $P(t,t+\tau_m)(\dot{}) - P(0,0+\tau_m)$. The zeroth to the fifth approximations for the ZC price change, as described in equations (48), are presented in the third column to the eight columns. The corresponding errors approximations are displayed in the appendix part in tables below in basis point. They represent the difference between the exact value and approximation from order 0 to 5.

It may be observed from these tables that the error is exactly equal to zero when there is no shock which means our approximation is exact for $\varepsilon = 0$.

Another main point is that the error decrease with the order of the approximation as showed in figures (4), (6),(8), (10), (12), (14), (16), (18), (20) where are ploted errors in basis point between the exact value and the approximation given the order.
These graphics also show us that is not necessary to go beyond the fifth order where all the errors reach an asymptote. We can explained that by the fact the nth term in our approximation is divided by a factorial n which tends to zero when the order grows. Then it becomes not relevant to take into account order larger than 5 since it will change our approximation less than 1.10^{-3} basis point our error.

As expected, when the shock moves away from zero our approximations are less efficient which is easily explained by the fact we have made a Taylor-Lagrange approximation.

In most of work the regression is generaly used since it also gives the sensitivities to any order with no any further computation. To compare the two methods we have ploted in figures (5), (7), (9),(11), (13), (15), (17), (19), (21) the errors obtained with our 5th order approximation (the green line) and the regression (the blue line). As one can notice, our approximation is way better especially for shocks away from 0. Our results show that our sensitivities give precise bond change approximations when compare to regression approach based only on data. It seems that the pain granted to our approach pays on accuracy.

4.2 CBB

As for the zero coupon price the illustrations are just limited to a short horizon $t = 15$ days and a CBB with a 100 dollar face value and a coupon rate of 0.1%. For table (**??**), in the first column is given some possible values of the shock. The exact CBB value change is displayed in the second column. Columns 3 to 8 reprensents the approximation of the CBB chnage value from order 0 to 5. The table below give the corresponding error in basis point.
With the shock $\varepsilon = 1.5$ the error approximation for the first order is $7.4\dot{1}0^{-3}$ which represent 0.0074% of the CBB face value. If instead of consediring just one CBB we deal with a position of 10 millions of such a CBB, then the error approximations becomes 74000. A loss with this magnitude size may be unacceptable for the hedger point of view. It means that limiting to the first order approximation should not be sufficient in prcatice. This is the reason why we also introduce and consider high order approximations. For example with the same shock $\varepsilon = 1.5$ but considering the fifth order approximation we get an error of $1.6\dot{1}0^{-3}$. It means that when considering a position of 10 millions of such a bond the error approximation is just reduced to 16000. Of course an amount loss with such a magnitude size is negligible for the hedger point of view.

5 Conclusion

1. Our derivation of the bond sensitivities relies on an approximation of the instantaneous forward as found by Krippner (2013) which, as highlighted by Christensen and Rudebusch (2015), should be viewed as not fully internally consistent and should be considered as simply an approximation to an arbitrage-free model. However the tractability benefit fully justifies our willing to base our findings with this Krippner's result. By the way, it would be interesting to better analyze the sensitivities inaccuracies spanned by our approach, when compared to results coming from Monte-Carlo simulations and Least-Square-Regressions.

2. We have obtained here global sensitivities in the sense that they have to apply without any restriction on the size of shocks affecting the underlying shadow rate. Some of our numerical experiments show that the resulting error approximations become high for shocks with large magnitudes. Careful inspection in our approach show that local results may actually be easily derived. This is particularly useful when one wishes to deal with extreme or stressed situations. The local sensitivities would provide better approximations than the direct ones we present in this paper.

3. When compared to regression the approximation we have performed for the zero coupon price change is way smaller in terms of error no matter the chosen shock or maturity.

4. Though the present work is only focused on the restrictive case, dealing with an underlying shadow rate driven by the one-factor Vasicek model (1-Vas), our results would highlight and provide a benchmark for further extensions, as the 1-Vas itself played with respect to classical term structure models. As seen here, derivation of sensitivities based on a shadow rate driven by a multi-dimension dynamic model would lead to technical complications.

5. Before investigating this high-dimensional situation, we think it is first useful to explore the sensitivities of related nonlinear interest rate products, as caps/floors.

6. In this paper, the model is already assumed to calibrated and our main focus is only on the derivation of the sensitivities with respect to the shock linked to the unobserved underlying shadow rate. The results obtained here may help to appreciate the stability of the model parameter used in the pricing for example.

7. Even for the time being negative yield-rates are present in markets and people expect gradual increase of the rates for the coming months, the studies of features related to the ZLB, as we have performed here, still deserve interest as they shed light on the alternative treatment toward to issues related to credit spreads, survival probabilities and volatilities, for which the associated processes are restricted to have nonnegative values.

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7 Appendix

7.1 Définition of the $a^*_{i,m}$ and $b_{i,m}$

We denote

$$a^*_{i,m}(0,n) = a\big(0,n;\lambda_{i,m},\nu_{i,m},\lambda^*_{i,m}\nu^*_{i,m}\big)$$

We set for $n=1$

$$a^*_{i,m}(0,1) \equiv -\lambda^*_{i,m}\nu^*_{i,m}, \quad a^*_{i,m}(1,1) \equiv -(\lambda^*_{i,m})^2$$

and for $n=2$

$$a^*_{i,m}(0;2) \equiv a^*_{i,m}(0;1)^2 + a^*_{i,m}(1;1),$$
$$a^*_{i,m}(1;2) \equiv 2a^*_{i,m}(0;1)a^*_{i,m}(1;1),$$
$$a^*_{i,m}(2;2) \equiv a^*_{i,m}(1;1)^2$$

then for $n \geq 3$, one has

$$a^*_{i,m}(0,n) = a^*_{i,m}(0,1)a^*_{i,m}(0,n-1) + a^*_{i,m}(1,n-1)$$

$$a^*_{i,m}(n-1,n) = a^*_{i,m}(0,1)a^*_{i,m}(n-1,n-1) + a^*_{i,m}(1,1)a^*_{i,m}(n-2,n-1)$$
$$a^*_{i,m}(n,n) = a^*_{i,m}(1,1)a^*_{i,m}(n-1,n-1)$$

For $1 \leq k \leq n-2$,

$$a^*_{i,m}(k,n) = a^*_{i,m}(0,1)a^*_{i,m}(k,n-1) + a^*_{i,m}(1,1)a^*_{i,m}(k-1,n-1) + (k+1)a^*_{i,m}(k+1,n-1).$$

The $b_{i,m}$'s are similarly defined by

$$b_{i,m}(0,n) = b\big(0, n; \lambda_{i,m}, \nu_{i,m}, \lambda^*_{i,m}, \nu^*_{i,m}\big).$$

$$b_{i,m}(0,2) \equiv \lambda^*_{i,m}\Big(2\lambda_{i,m} + \nu a^*_{i,m}(0;1)\Big),$$

$$b_{i,m}(1,2) \equiv \lambda^*_{i,m}\Big(\nu a^*_{i,m}(1;1) + \lambda_{i,m} a^*_{i,m}(0;1)\Big)$$

$$b_{i,m}(2,2) \equiv \lambda^*_{i,m}\lambda_{i,m} a^*_{i,m}(1;1)$$

for $n \geq 3$, one has

$$b_{i,m}(0,n) = a^*_{i,m}(0,1) b_{i,m}(0,n-1) + b_{i,m}(1,n-1)$$

$$b_{i,m}(n-1,n) = a^*_{i,m}(0,1) b_{i,m}(n-1,n-1) + a^*_{i,m}(1,1) b_{i,m}(n-2,n-1)$$

$$b_{i,m}(n,n) = a^*_{i,m}(1,1) b_{i,m}(n-1,n-1)$$

for $\quad 1 \leq k \leq n-2$, one has

$$b_{i,m}(k,n) = a^*_{i,m}(0,1) b_{i,m}(k,n-1) + a^*_{i,m}(1,1) b_{i,m}(k-1,n-1) + (k+1) b_{i,m}(k+1,n-1).$$

7.2 Tables and plots for the Zero-coupon-bond price change approximation

maturity (years)	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5

Set calibration

t (Days)	x_0 (%)	κ (%)	σ (%)	θ (%)
15	2.5	13.2103	2.3520	6.8720

7.2.1 Comparaison of approximations errors

In the following we want to show, with plots and tables, how the error between the real ZCB price change and our approximation evolve when the order of the performed approximation increase.

For each maturity, we first give the plot for some given shock in $(-1.5, 1.5)$, of these erros. Then the corresponding tables are given below each figure so that the reader get more precision. Finally we give the plot of the 5th order error in function of the shock and make a comparison with the error found with a linear regression.
Note that all the results are given in basis point.

- For $\tau_m = 1$: When the maturity is 1 year, our approximation gets better for negative shocks when the order increase, however for positive shocks, the order 3 give us the smallest error.

Figure 4: $m = 1$ year

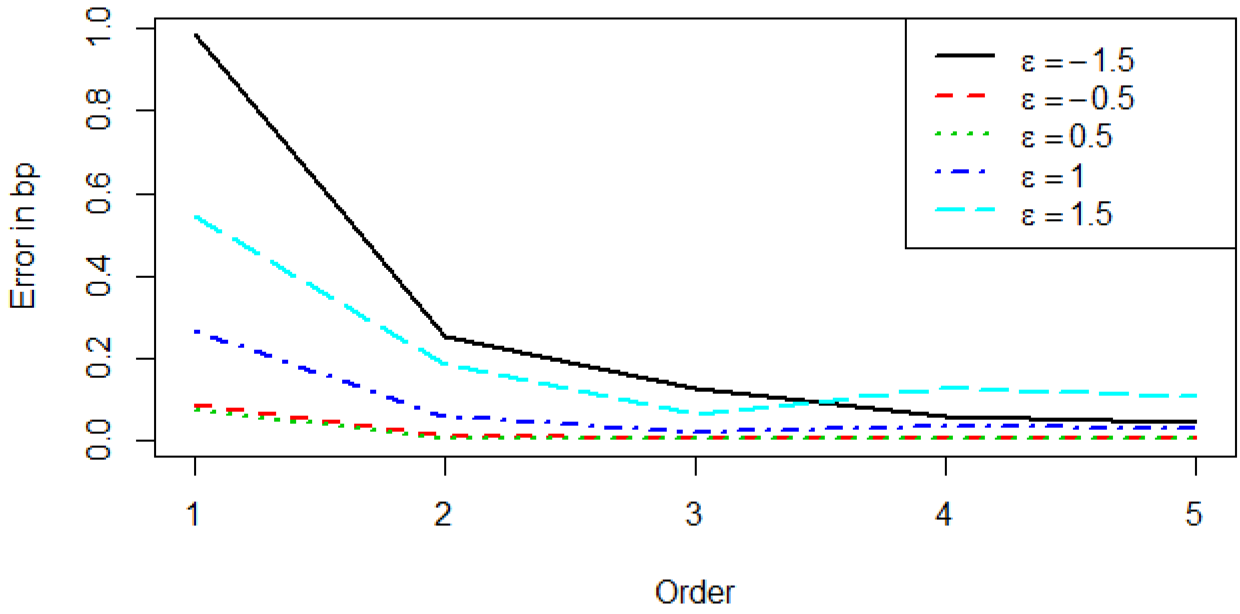
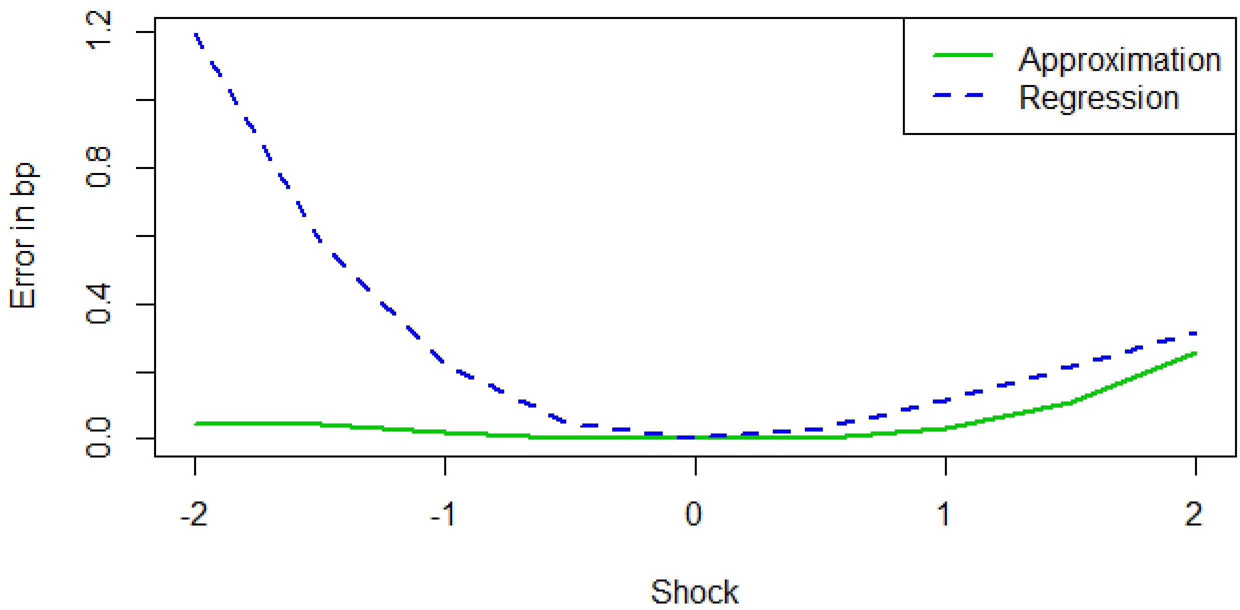


Table 1: $m = 1$ year

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	112,9447	117,6770	114,9888	114,1025	113,2127	112,7293
-3,0	97,4633	100,6685	98,6935	98,1354	97,6551	97,4314
-2,5	81,6078	83,6601	82,2885	81,9656	81,7339	81,6440
-2,0	65,4404	66,6516	65,7738	65,6085	65,5136	65,4841
-1,5	49,0147	49,6431	49,1494	49,0796	49,0496	49,0426
-1,0	32,3770	32,6347	32,4152	32,3946	32,3886	32,3877
-0,5	15,5668	15,6262	15,5713	15,5688	15,5684	15,5684
0,0	-1,3823	-1,3823	-1,3823	-1,3823	-1,3823	-1,3823
0,5	-18,4414	-18,3907	-18,4456	-18,4430	-18,4434	-18,4433
1,0	-35,5863	-35,3992	-35,6186	-35,5980	-35,6039	-35,6030
1,5	-52,7964	-52,4077	-52,9014	-52,8316	-52,8617	-52,8547
2,0	-70,0545	-69,4161	-70,2939	-70,1285	-70,2234	-70,1940
2,5	-87,3462	-86,4246	-87,7961	-87,4731	-87,7048	-87,6149
3,0	-104,6594	-103,4331	-105,4080	-104,8499	-105,3303	-105,1066
3,5	-121,9841	-120,4415	-123,1297	-122,2434	-123,1333	-122,6499

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	4,7323	2,0441	1,1579	0,2680	0,2154
-3,0	3,2052	1,2302	0,6721	0,1918	0,0319
-2,5	2,0523	0,6808	0,3578	0,1261	0,0363
-2,0	1,2112	0,3335	0,1681	0,0732	0,0438
-1,5	0,6284	0,1347	0,0649	0,0349	0,0279
-1,0	0,2577	0,0382	0,0176	0,0116	0,0107
-0,5	0,0594	0,0046	0,0020	0,0016	0,0016
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0506	0,0042	0,0016	0,0020	0,0020
1,0	0,1871	0,0324	0,0117	0,0176	0,0167
1,5	0,3887	0,1050	0,0353	0,0653	0,0583
2,0	0,6384	0,2394	0,0740	0,1689	0,1394
2,5	0,9216	0,4499	0,1269	0,3586	0,2687
3,0	1,2263	0,7487	0,1905	0,6709	0,4472
3,5	1,5426	1,1456	0,2593	1,1492	0,6658

Figure 5: $m = 1$ year



- For $\tau_m = 1.5$ years : We observe the same results as for the 1 year maturity.

Figure 6: $m = 1.5$ years

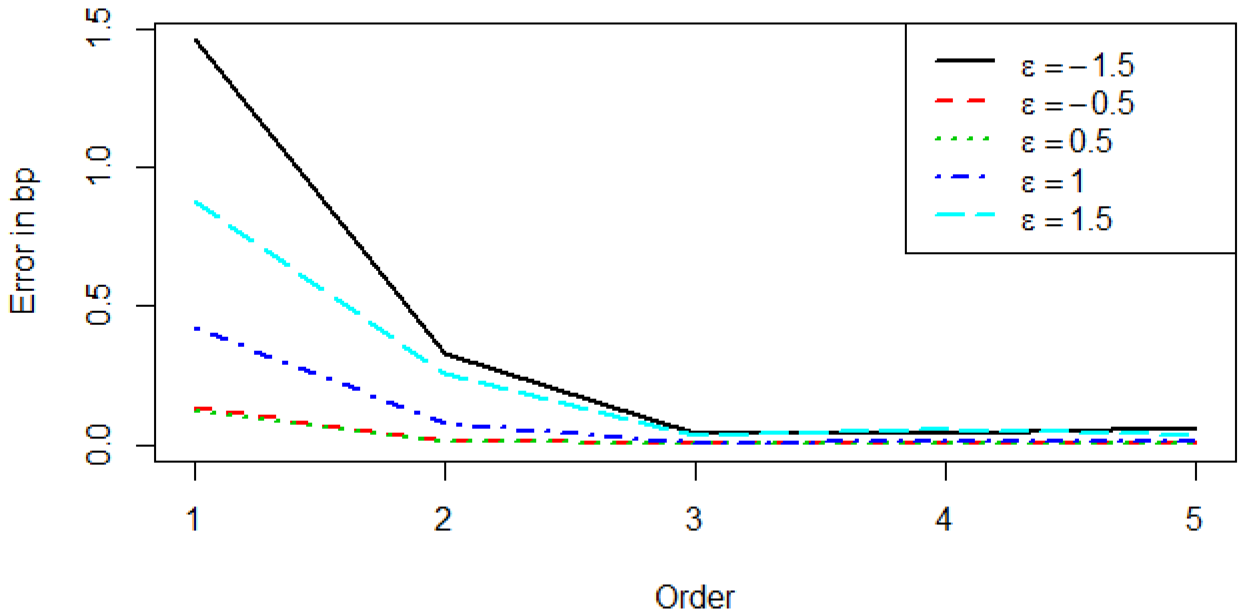
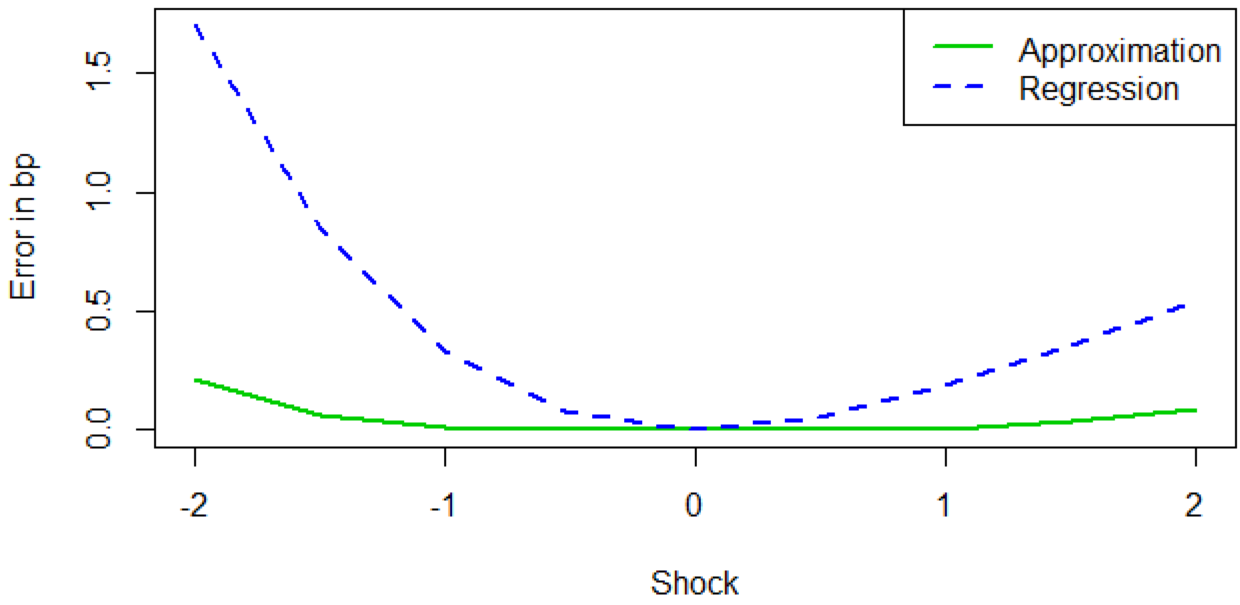


Table 2: $m = 1.5$ year

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	158,2769	165,0093	160,8485	158,8467	157,7308	157,2424
-3,0	136,5424	141,1597	138,1027	136,8421	136,2398	136,0139
-2,5	114,3167	117,3100	115,1871	114,4576	114,1672	114,0764
-2,0	91,6719	93,4604	92,1017	91,7282	91,6093	91,5795
-1,5	68,6715	69,6107	68,8465	68,6889	68,6513	68,6442
-1,0	45,3714	45,7611	45,4214	45,3748	45,3673	45,3664
-0,5	21,8205	21,9115	21,8265	21,8207	21,8202	21,8202
0,0	-1,9382	-1,9382	-1,9382	-1,9382	-1,9382	-1,9382
0,5	-25,8671	-25,7878	-25,8727	-25,8669	-25,8674	-25,8673
1,0	-49,9334	-49,6375	-49,9771	-49,9304	-49,9379	-49,9369
1,5	-74,1086	-73,4871	-74,2513	-74,0938	-74,1314	-74,1244
2,0	-98,3678	-97,3367	-98,6954	-98,3219	-98,4409	-98,4111
2,5	-122,6895	-121,1864	-123,3093	-122,5798	-122,8703	-122,7794
3,0	-147,0549	-145,0360	-148,0930	-146,8324	-147,4347	-147,2088
3,5	-171,4480	-168,8857	-173,0465	-171,0448	-172,1607	-171,6723

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	6,7325	2,5716	0,5699	0,5461	1,0344
-3,0	4,6172	1,5603	0,2997	0,3026	0,5286
-2,5	2,9933	0,8704	0,1409	0,1496	0,2404
-2,0	1,7884	0,4298	0,0563	0,0627	0,0924
-1,5	0,9392	0,1750	0,0174	0,0203	0,0273
-1,0	0,3897	0,0500	0,0034	0,0041	0,0050
-0,5	0,0910	0,0060	0,0002	0,0003	0,0003
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0793	0,0056	0,0002	0,0003	0,0002
1,0	0,2960	0,0437	0,0030	0,0044	0,0035
1,5	0,6215	0,1427	0,0149	0,0228	0,0157
2,0	1,0311	0,3276	0,0459	0,0730	0,0433
2,5	1,5031	0,6198	0,1097	0,1808	0,0900
3,0	2,0189	1,0381	0,2225	0,3798	0,1539
3,5	2,5624	1,5985	0,4033	0,7127	0,2243

Figure 7: $m = 1.5$ years



- For $\tau_m = 2$ years the tendency is reversed. Indeed, we now observes for positives shocks a continuous decrease with the order when for negative shocks, the order 3 becomes the best.

Figure 8: $m = 2$ years

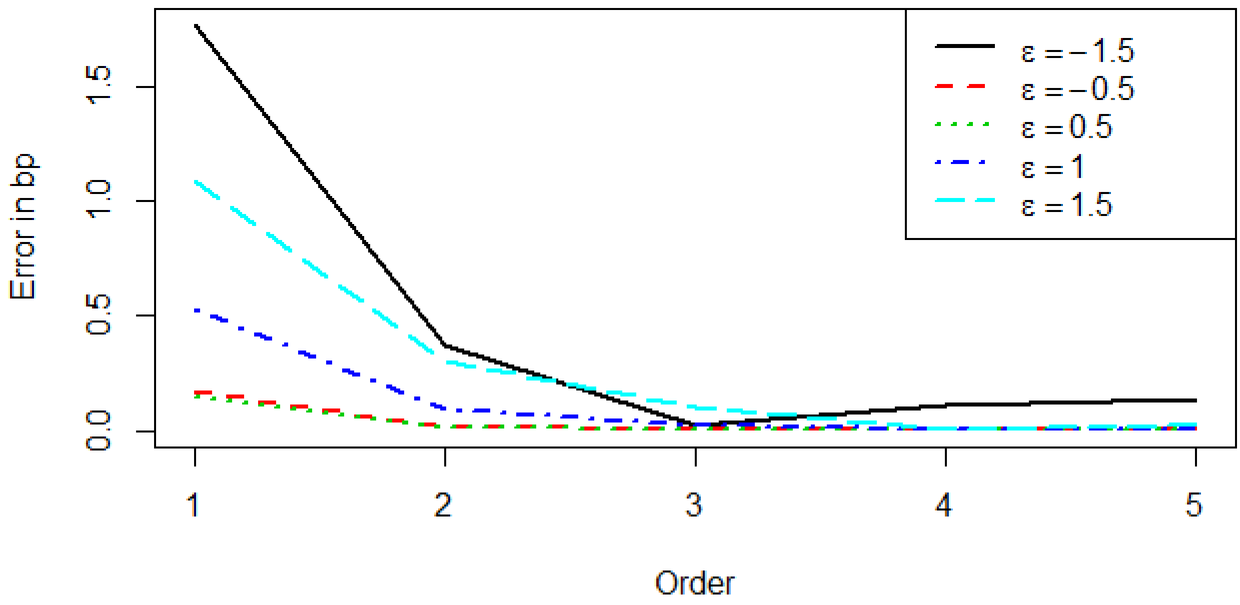
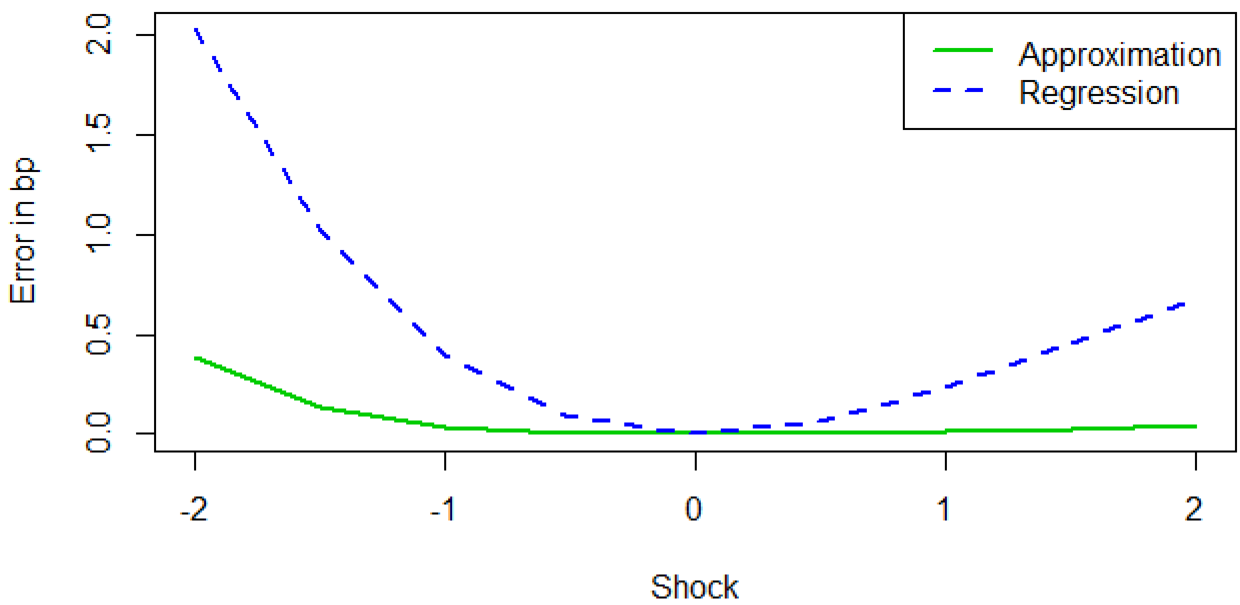


Table 3: $m = 2$ year

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	198,5518	206,5639	201,4616	198,6750	197,4544	196,9704
-3,0	171,1873	176,7081	172,9596	171,2047	170,5459	170,3219
-2,5	143,2568	146,8524	144,2492	143,2337	142,9160	142,8260
-2,0	114,8387	116,9967	115,3306	114,8107	114,6805	114,6510
-1,5	86,0028	87,1409	86,2038	85,9844	85,9432	85,9362
-1,0	56,8110	57,2852	56,8687	56,8037	56,7955	56,7946
-0,5	27,3183	27,4294	27,3253	27,3172	27,3167	27,3167
0,0	-2,4263	-2,4263	-2,4263	-2,4263	-2,4263	-2,4263
0,5	-32,3796	-32,2820	-32,3862	-32,3780	-32,3785	-32,3785
1,0	-62,5032	-62,1378	-62,5543	-62,4893	-62,4974	-62,4965
1,5	-92,7631	-91,9935	-92,9306	-92,7113	-92,7525	-92,7455
2,0	-123,1294	-121,8492	-123,5153	-122,9953	-123,1255	-123,0960
2,5	-153,5756	-151,7050	-154,3082	-153,2926	-153,6104	-153,5204
3,0	-184,0783	-181,5607	-185,3093	-183,5545	-184,2133	-183,9893
3,5	-214,6171	-211,4165	-216,5187	-213,7321	-214,9527	-214,4686

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	8,0121	2,9098	0,1232	1,0974	1,5814
-3,0	5,5209	1,7723	0,0174	0,6414	0,8654
-2,5	3,5956	0,9924	0,0231	0,3408	0,4308
-2,0	2,1580	0,4919	0,0280	0,1582	0,1877
-1,5	1,1381	0,2010	0,0184	0,0595	0,0665
-1,0	0,4742	0,0577	0,0073	0,0154	0,0163
-0,5	0,1111	0,0070	0,0011	0,0016	0,0017
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0975	0,0066	0,0015	0,0010	0,0011
1,0	0,3654	0,0511	0,0139	0,0057	0,0067
1,5	0,7696	0,1675	0,0518	0,0106	0,0176
2,0	1,2802	0,3859	0,1341	0,0039	0,0334
2,5	1,8706	0,7325	0,2830	0,0348	0,0553
3,0	2,5176	1,2310	0,5239	0,1350	0,0890
3,5	3,2006	1,9016	0,8850	0,3356	0,1485

Figure 9: $m = 2$ years



- For $\tau_m = 2.5$ years

Figure 10: $m = 2.5$ years

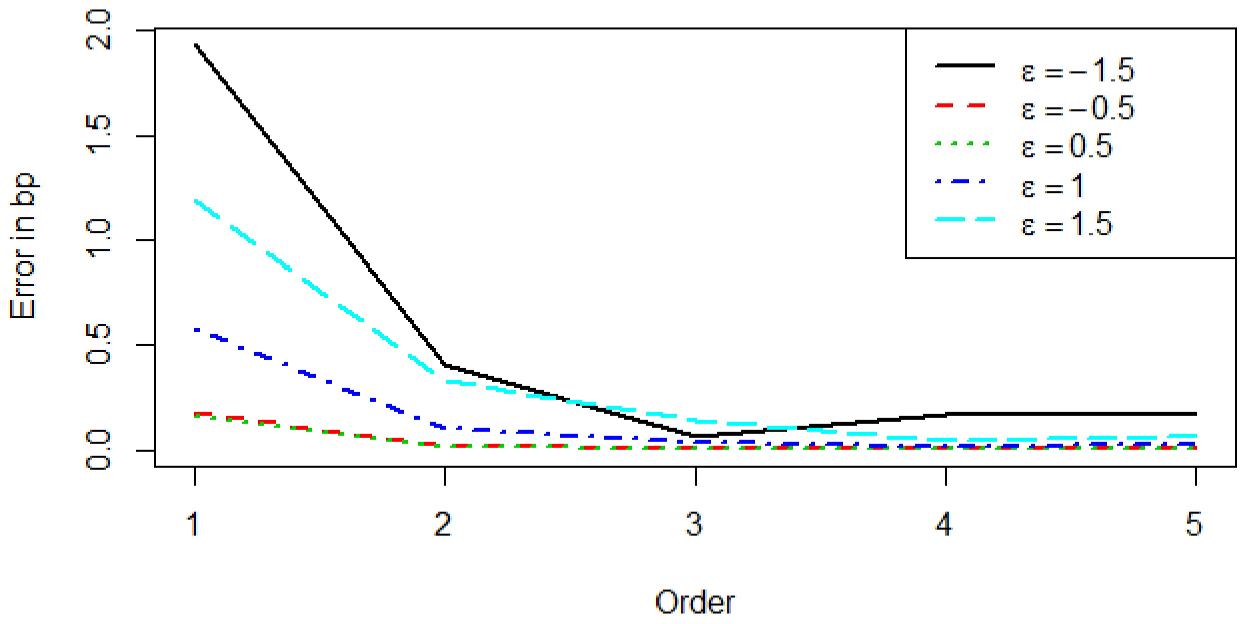
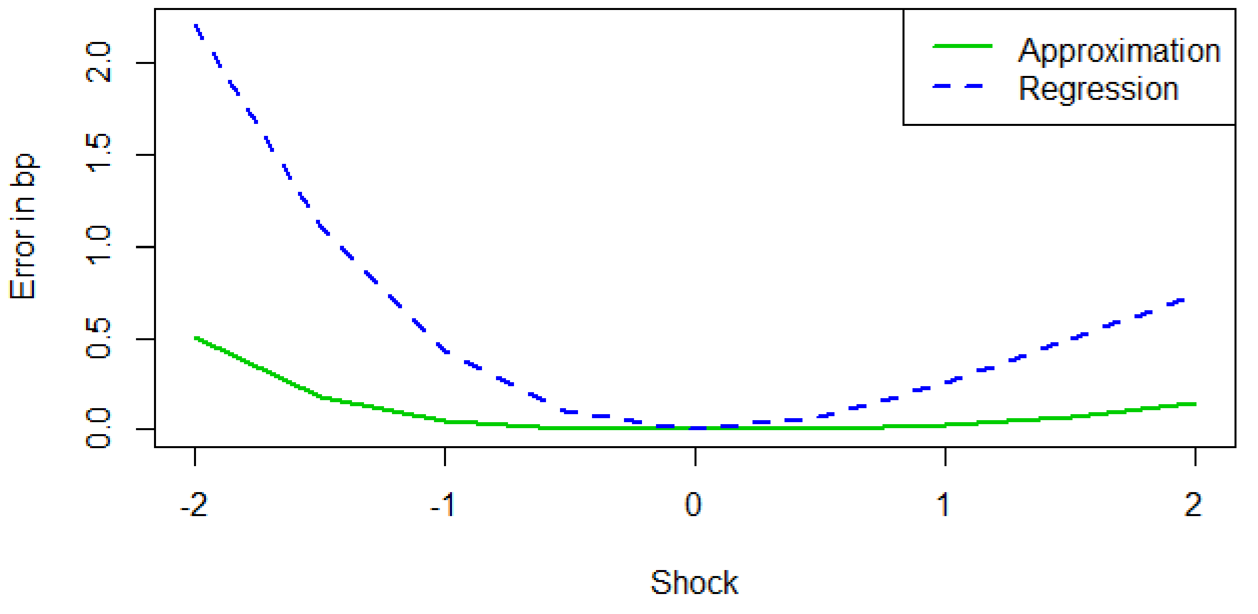


Table 4: $m = 2.5$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	234,3792	243,1086	237,5184	234,1840	232,9088	232,4301
-3,0	201,9477	207,9709	203,8638	201,7640	201,0757	200,8542
-2,5	168,9057	172,8331	169,9810	168,7658	168,4339	168,3449
-2,0	135,3359	137,6954	135,8700	135,2479	135,1119	135,0827
-1,5	101,3122	102,5576	101,5309	101,2684	101,2254	101,2184
-1,0	66,9006	67,4199	66,9635	66,8858	66,8773	66,8764
-0,5	32,1604	32,2821	32,1681	32,1583	32,1578	32,1578
0,0	-2,8556	-2,8556	-2,8556	-2,8556	-2,8556	-2,8556
0,5	-38,1002	-37,9933	-38,1074	-38,0977	-38,0982	-38,0982
1,0	-73,5312	-73,1311	-73,5874	-73,5097	-73,5182	-73,5172
1,5	-109,1111	-108,2688	-109,2956	-109,0331	-109,0762	-109,0692
2,0	-144,8061	-143,4066	-145,2319	-144,6098	-144,7458	-144,7166
2,5	-180,5865	-178,5443	-181,3965	-180,1813	-180,5133	-180,4242
3,0	-216,4255	-213,6821	-217,7891	-215,6893	-216,3777	-216,1562
3,5	-252,2995	-248,8198	-254,4100	-251,0756	-252,3508	-251,8721

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	8,7294	3,1392	0,1952	1,4704	1,9492
-3,0	6,0232	1,9161	0,1837	0,8721	1,0935
-2,5	3,9274	1,0753	0,1399	0,4719	0,5609
-2,0	2,3595	0,5341	0,0881	0,2240	0,2532
-1,5	1,2455	0,2187	0,0438	0,0868	0,0937
-1,0	0,5193	0,0629	0,0148	0,0233	0,0243
-0,5	0,1217	0,0076	0,0021	0,0026	0,0026
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,1069	0,0072	0,0025	0,0020	0,0020
1,0	0,4002	0,0562	0,0216	0,0131	0,0140
1,5	0,8422	0,1845	0,0779	0,0349	0,0418
2,0	1,3995	0,4258	0,1963	0,0604	0,0895
2,5	2,0421	0,8100	0,4052	0,0732	0,1622
3,0	2,7434	1,3637	0,7361	0,0478	0,2693
3,5	3,4797	2,1105	1,2239	0,0514	0,4274

Figure 11: $m = 2.5$ year



- For $\tau_m = 3$ years

Figure 12: $m = 3$ years

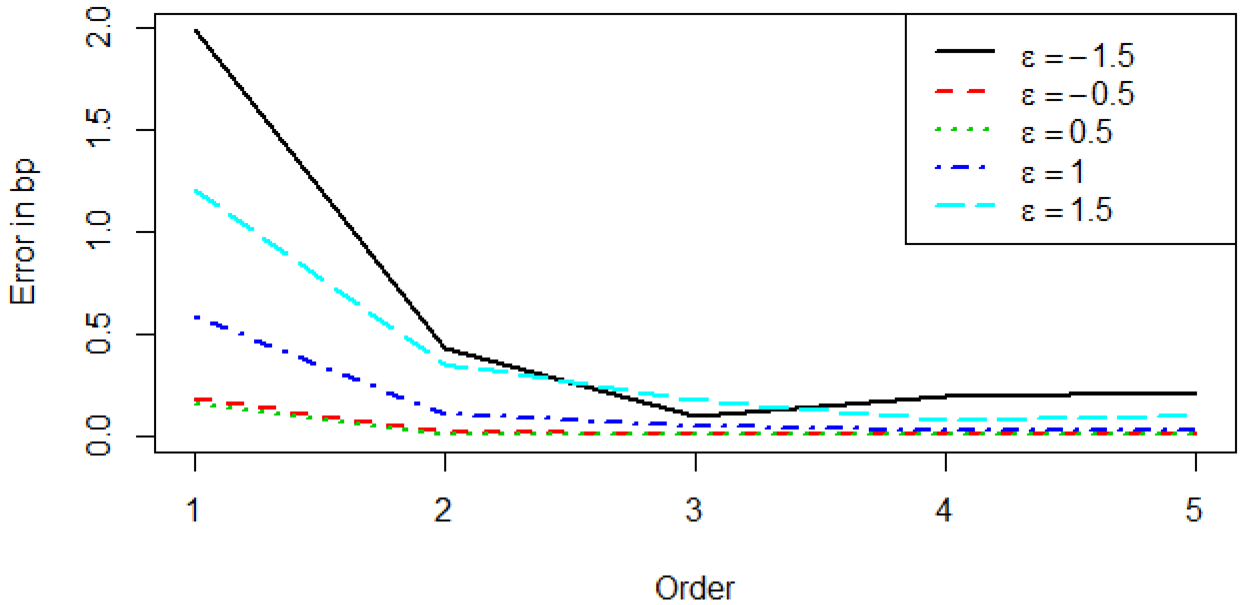
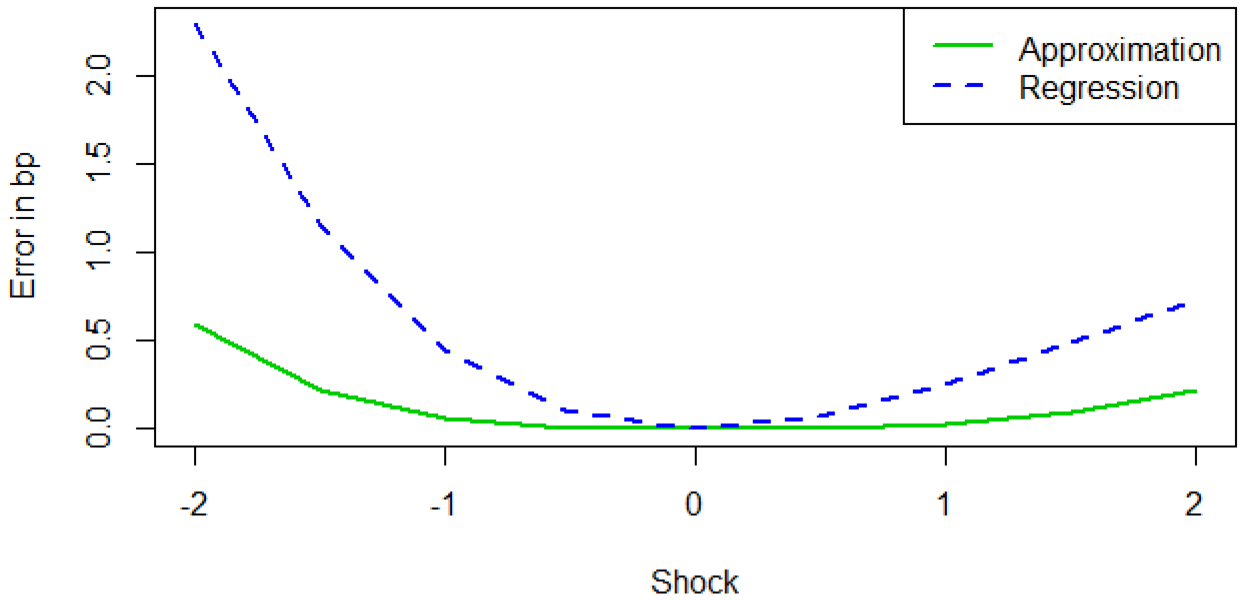


Table 5: $m = 3$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	266,1659	275,1839	269,4635	265,7463	264,4415	263,9683
-3,0	229,1920	235,4102	231,2074	228,8665	228,1622	227,9433
-2,5	191,5853	195,6364	192,7178	191,3631	191,0235	190,9355
-2,0	153,4315	155,8626	153,9947	153,3011	153,1620	153,1332
-1,5	114,8072	116,0889	115,0382	114,7456	114,7015	114,6947
-1,0	75,7816	76,3151	75,8481	75,7614	75,7527	75,7518
-0,5	36,4165	36,5413	36,4246	36,4137	36,4132	36,4132
0,0	-3,2324	-3,2324	-3,2324	-3,2324	-3,2324	-3,2324
0,5	-43,1153	-43,0062	-43,1230	-43,1121	-43,1127	-43,1126
1,0	-83,1873	-82,7800	-83,2470	-83,1603	-83,1690	-83,1681
1,5	-123,4081	-122,5538	-123,6045	-123,3118	-123,3559	-123,3490
2,0	-163,7418	-162,3275	-164,1954	-163,5018	-163,6410	-163,6121
2,5	-204,1561	-202,1013	-205,0199	-203,6652	-204,0049	-203,9169
3,0	-244,6219	-241,8751	-246,0778	-243,7370	-244,4413	-244,2224
3,5	-285,1133	-281,6488	-287,3693	-283,6521	-284,9569	-284,4837

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	9,0181	3,2976	0,4196	1,7244	2,1976
-3,0	6,2182	2,0154	0,3254	1,0297	1,2487
-2,5	4,0511	1,1325	0,2222	0,5618	0,6498
-2,0	2,4312	0,5633	0,1303	0,2695	0,2983
-1,5	1,2816	0,2309	0,0617	0,1057	0,1125
-1,0	0,5335	0,0665	0,0202	0,0289	0,0298
-0,5	0,1248	0,0081	0,0027	0,0033	0,0033
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,1091	0,0077	0,0032	0,0026	0,0027
1,0	0,4073	0,0597	0,0270	0,0183	0,0192
1,5	0,8544	0,1963	0,0963	0,0523	0,0591
2,0	1,4143	0,4536	0,2400	0,1009	0,1297
2,5	2,0548	0,8638	0,4909	0,1512	0,2392
3,0	2,7469	1,4559	0,8849	0,1806	0,3996
3,5	3,4645	2,2559	1,4612	0,1564	0,6297

Figure 13: $m = 3$ year



- For $\tau_m = 3.5$ years

Figure 14: $m = 3.5$ years

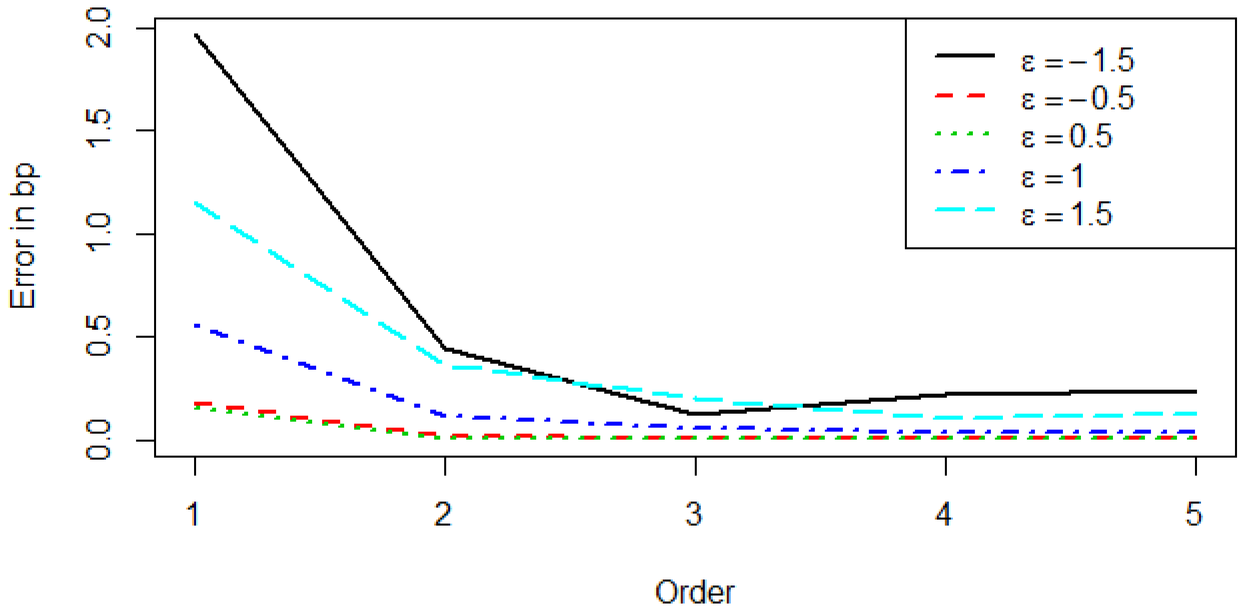
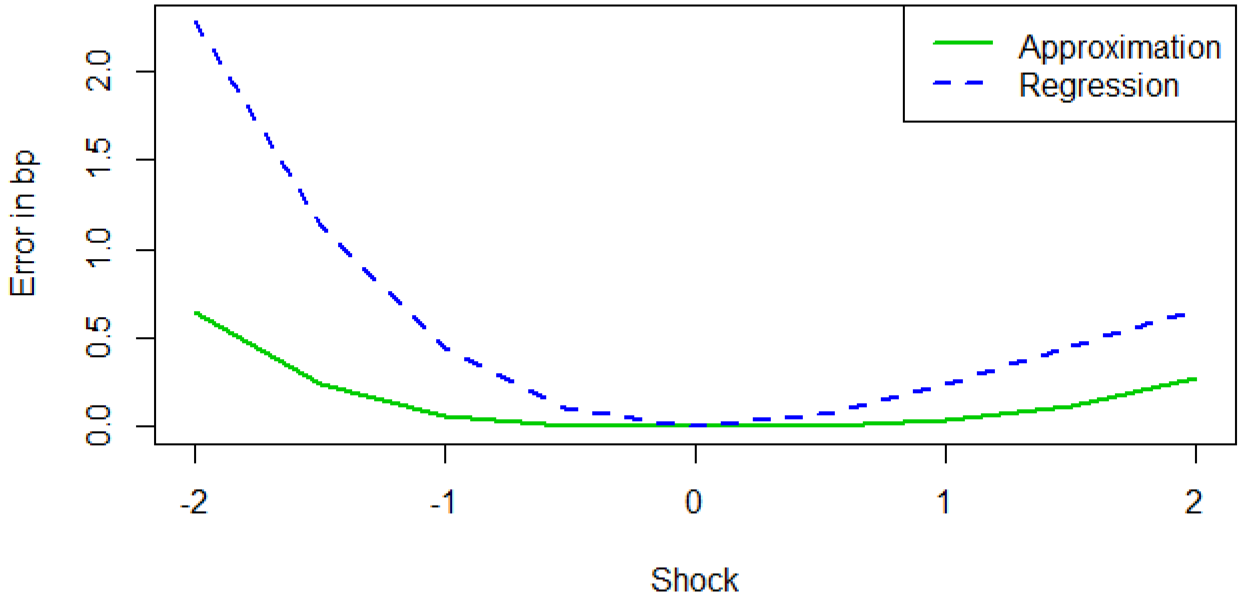


Table 6: $m = 3.5$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	294,2360	303,2191	297,6417	293,6592	292,3394	291,8718
-3,0	253,2124	259,3932	255,2955	252,7877	252,0752	251,8589
-2,5	211,5502	215,5674	212,7218	211,2704	210,9269	210,8399
-2,0	169,3372	171,7415	169,9203	169,1773	169,0365	169,0080
-1,5	126,6520	127,9157	126,8913	126,5778	126,5332	126,5265
-1,0	83,5655	84,0898	83,6345	83,5416	83,5329	83,5320
-0,5	40,1418	40,2640	40,1502	40,1385	40,1380	40,1380
0,0	-3,5619	-3,5619	-3,5619	-3,5619	-3,5619	-3,5619
0,5	-47,4936	-47,3877	-47,5015	-47,4899	-47,4905	-47,4904
1,0	-91,6067	-91,2135	-91,6688	-91,5760	-91,5848	-91,5839
1,5	-135,8594	-135,0394	-136,0638	-135,7503	-135,7948	-135,7881
2,0	-180,2138	-178,8652	-180,6864	-179,9433	-180,0841	-180,0556
2,5	-224,6360	-222,6911	-225,5367	-224,0854	-224,4289	-224,3420
3,0	-269,0953	-266,5169	-270,6146	-268,1067	-268,8191	-268,6028
3,5	-313,5643	-310,3428	-315,9201	-311,9377	-313,2575	-312,7899

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	8,9830	3,4057	0,5768	1,8966	2,3642
-3,0	6,1809	2,0832	0,4247	1,1371	1,3535
-2,5	4,0171	1,1715	0,2798	0,6234	0,7103
-2,0	2,4043	0,5832	0,1599	0,3006	0,3291
-1,5	1,2637	0,2393	0,0742	0,1187	0,1255
-1,0	0,5243	0,0690	0,0239	0,0327	0,0336
-0,5	0,1222	0,0084	0,0032	0,0038	0,0038
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,1059	0,0080	0,0036	0,0031	0,0031
1,0	0,3932	0,0621	0,0308	0,0220	0,0229
1,5	0,8200	0,2044	0,1091	0,0646	0,0713
2,0	1,3486	0,4726	0,2705	0,1297	0,1582
2,5	1,9449	0,9007	0,5506	0,2070	0,2940
3,0	2,5783	1,5193	0,9886	0,2762	0,4925
3,5	3,2215	2,3559	1,6265	0,3067	0,7743

Figure 15: $m = 3.5$ year



- For $\tau_m = 4$ years

Figure 16: $m = 4$ years

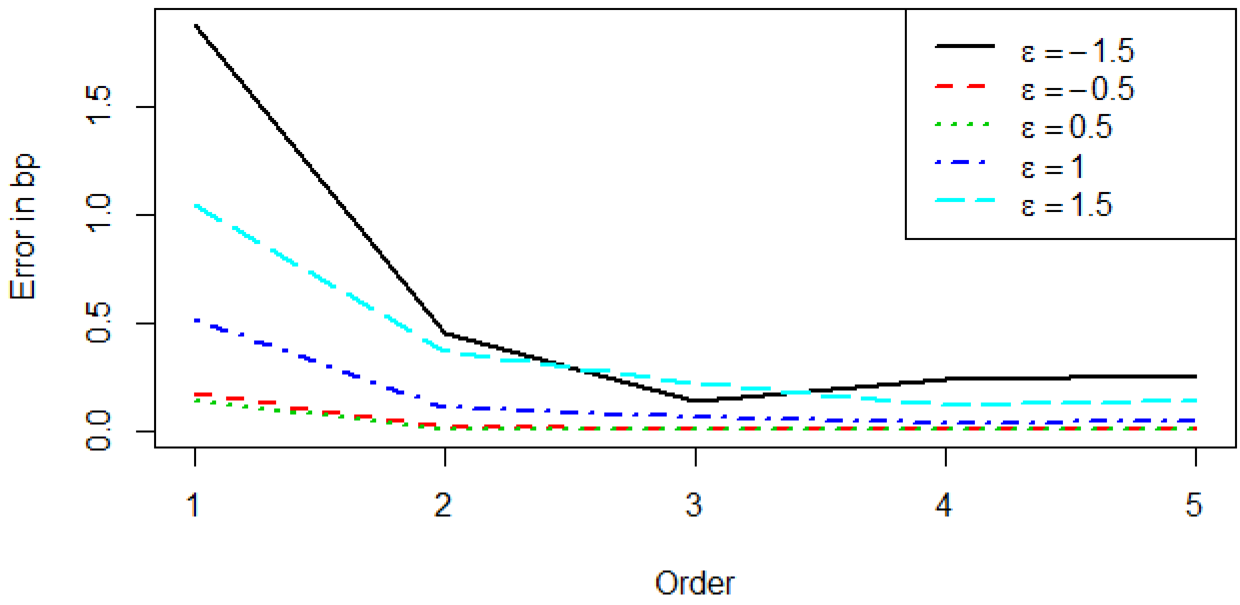
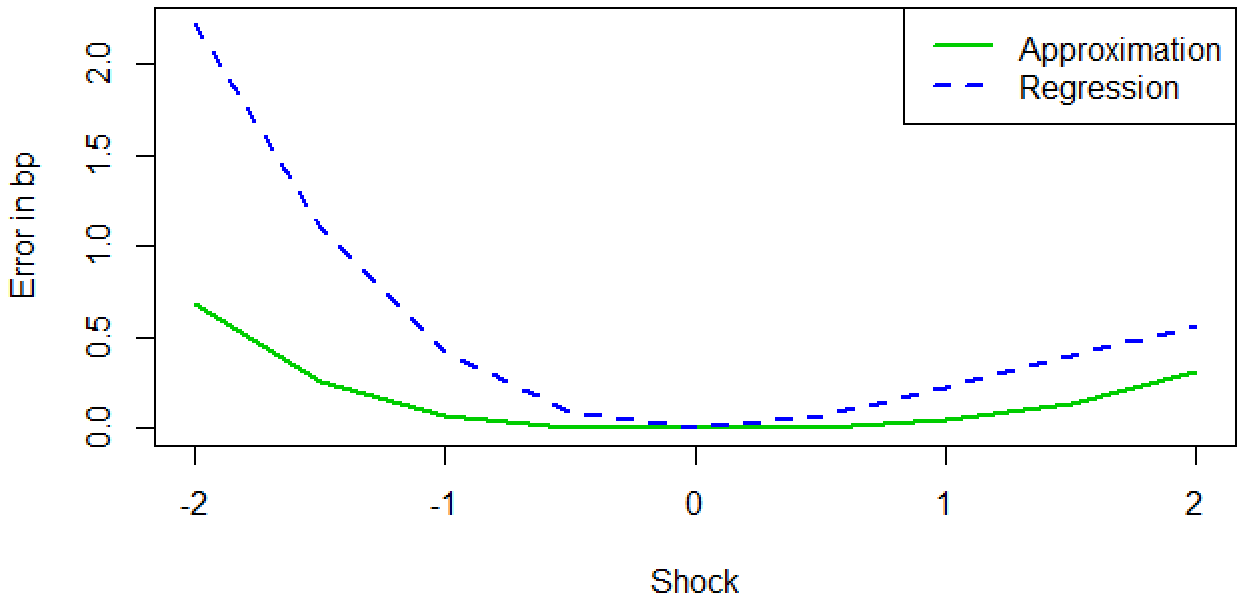


Table 7: $m = 4$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	318,8741	327,5806	322,3501	318,1883	316,8629	316,4011
-3,0	274,2635	280,2336	276,3909	273,7700	273,0546	272,8409
-2,5	229,0210	232,8867	230,2181	228,7014	228,3564	228,2705
-2,0	183,2356	185,5397	183,8318	183,0553	182,9139	182,8858
-1,5	136,9873	138,1927	137,2321	136,9044	136,8597	136,8531
-1,0	90,3482	90,8458	90,4188	90,3217	90,3129	90,3120
-0,5	43,3835	43,4988	43,3921	43,3799	43,3794	43,3794
0,0	-3,8481	-3,8481	-3,8481	-3,8481	-3,8481	-3,8481
0,5	-51,2937	-51,1951	-51,3018	-51,2897	-51,2903	-51,2902
1,0	-98,9053	-98,5421	-98,9690	-98,8720	-98,8808	-98,8799
1,5	-146,6400	-145,8890	-146,8497	-146,5221	-146,5668	-146,5602
2,0	-194,4587	-193,2360	-194,9439	-194,1673	-194,3087	-194,2805
2,5	-242,3265	-240,5830	-243,2515	-241,7349	-242,0799	-241,9940
3,0	-290,2116	-287,9299	-291,7727	-289,1518	-289,8673	-289,6537
3,5	-338,0855	-335,2769	-340,5073	-336,3455	-337,6710	-337,2092

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	8,7065	3,4761	0,6858	2,0112	2,4729
-3,0	5,9702	2,1274	0,4935	1,2089	1,4225
-2,5	3,8656	1,1970	0,3197	0,6647	0,7505
-2,0	2,3041	0,5962	0,1804	0,3217	0,3498
-1,5	1,2055	0,2448	0,0828	0,1276	0,1342
-1,0	0,4976	0,0706	0,0265	0,0353	0,0362
-0,5	0,1153	0,0086	0,0035	0,0041	0,0041
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0986	0,0082	0,0040	0,0034	0,0034
1,0	0,3633	0,0637	0,0334	0,0245	0,0254
1,5	0,7510	0,2097	0,1179	0,0732	0,0798
2,0	1,2228	0,4852	0,2914	0,1501	0,1782
2,5	1,7435	0,9251	0,5916	0,2466	0,3325
3,0	2,2817	1,5611	1,0597	0,3443	0,5579
3,5	2,8086	2,4219	1,7400	0,4145	0,8763

Figure 17: $m = 4$ years



- For $\tau_m = 4.5$ years

Figure 18: $m = 4.5$ years

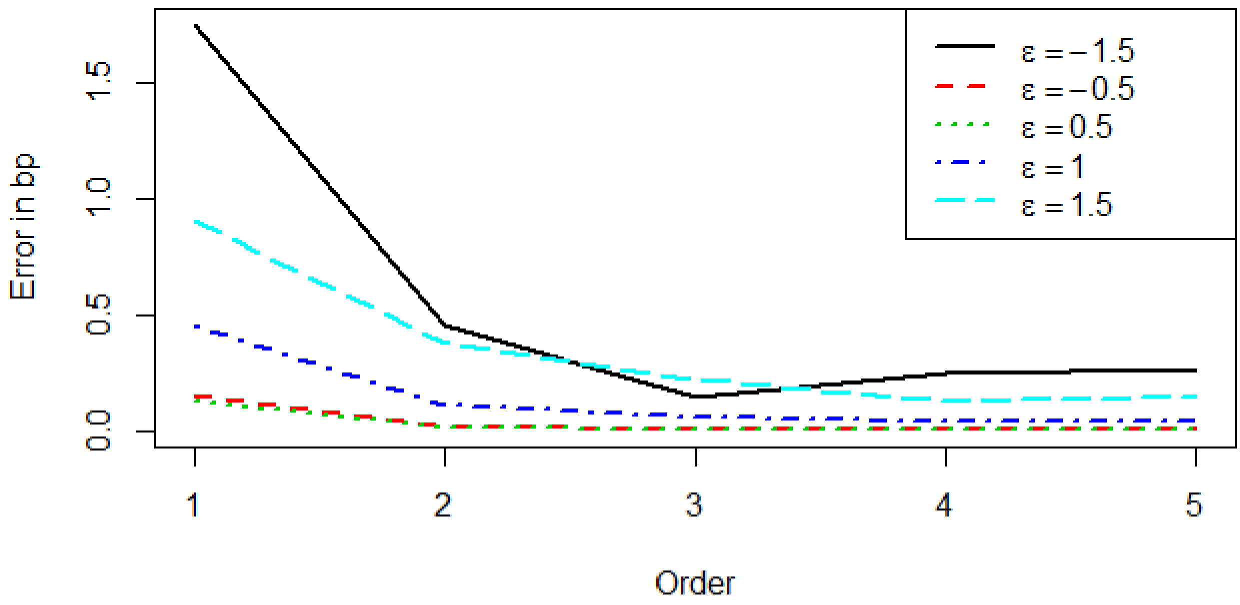
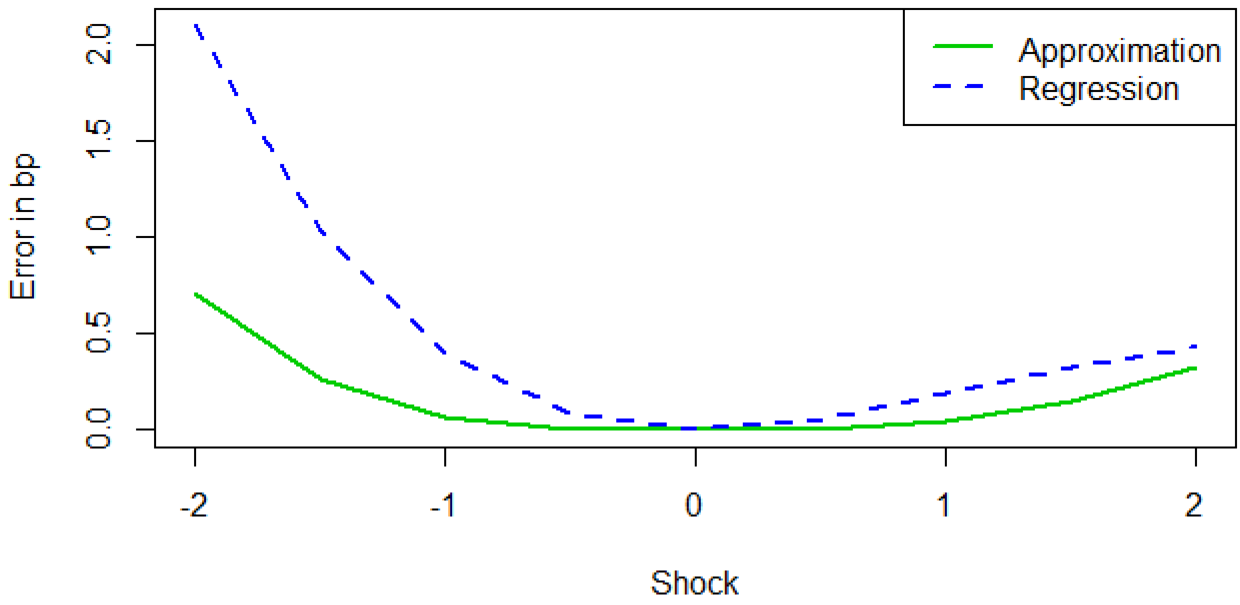


Table 8: $m = 4.5$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	340,3418	348,5948	343,8590	339,5820	338,2575	337,8018
-3,0	292,5778	298,2105	294,7311	292,0377	291,3228	291,1120
-2,5	244,1980	247,8262	245,4100	243,8513	243,5066	243,4218
-2,0	195,2917	197,4420	195,8956	195,0975	194,9563	194,9286
-1,5	145,9399	147,0577	146,1879	145,8512	145,8065	145,7999
-1,0	96,2153	96,6734	96,2868	96,1871	96,1783	96,1774
-0,5	46,1838	46,2892	46,1925	46,1800	46,1795	46,1795
0,0	-4,0951	-4,0951	-4,0951	-4,0951	-4,0951	-4,0951
0,5	-54,5677	-54,4794	-54,5760	-54,5636	-54,5641	-54,5641
1,0	-105,1856	-104,8636	-105,2502	-105,1505	-105,1593	-105,1584
1,5	-155,9049	-155,2479	-156,1178	-155,7811	-155,8258	-155,8192
2,0	-206,6859	-205,6322	-207,1786	-206,3805	-206,5218	-206,4940
2,5	-257,4930	-256,0165	-258,4327	-256,8740	-257,2188	-257,1341
3,0	-308,2938	-306,4007	-309,8801	-307,1867	-307,9016	-307,6908
3,5	-359,0592	-356,7850	-361,5208	-357,2438	-358,5683	-358,1127

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	8,2530	3,5172	0,7598	2,0843	2,5399
-3,0	5,6327	2,1533	0,5401	1,2550	1,4658
-2,5	3,6283	1,2120	0,3467	0,6914	0,7761
-2,0	2,1502	0,6038	0,1942	0,3354	0,3632
-1,5	1,1178	0,2480	0,0887	0,1334	0,1399
-1,0	0,4582	0,0716	0,0282	0,0370	0,0379
-0,5	0,1054	0,0087	0,0038	0,0043	0,0043
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0884	0,0083	0,0042	0,0036	0,0037
1,0	0,3220	0,0646	0,0351	0,0263	0,0272
1,5	0,6569	0,2129	0,1238	0,0791	0,0857
2,0	1,0537	0,4927	0,3054	0,1642	0,1919
2,5	1,4765	0,9397	0,6190	0,2742	0,3589
3,0	1,8931	1,5863	1,1071	0,3922	0,6030
3,5	2,2742	2,4616	1,8154	0,4909	0,9465

Figure 19: $m = .5$ years



- For $\tau_m = 5$ years

Figure 20: $m = 5$ years

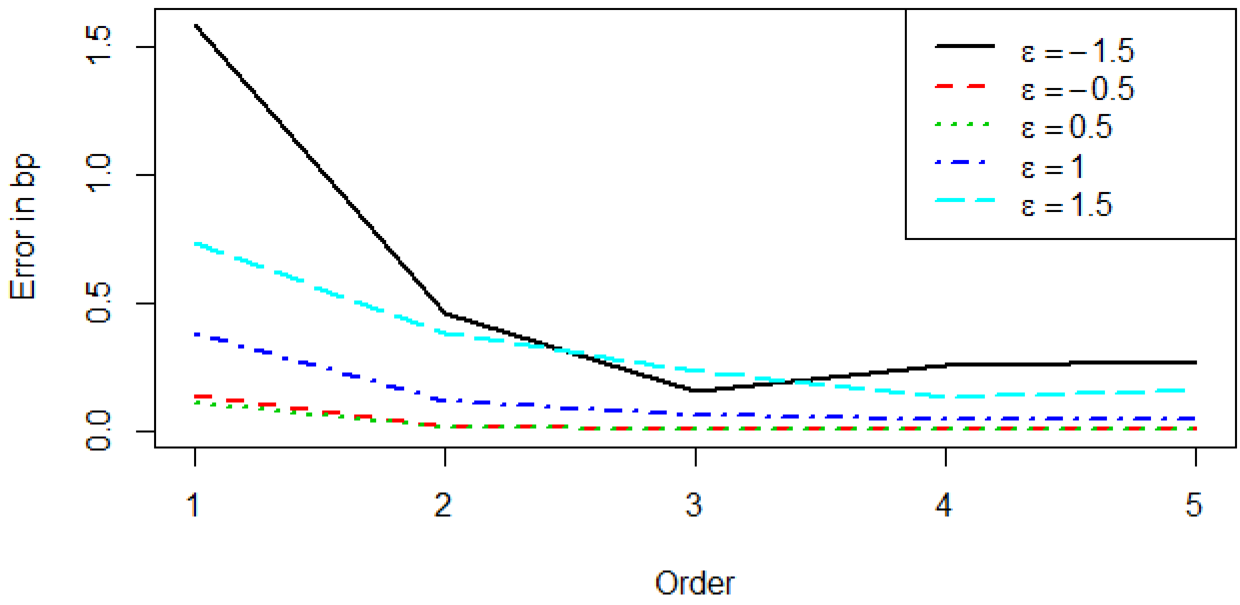
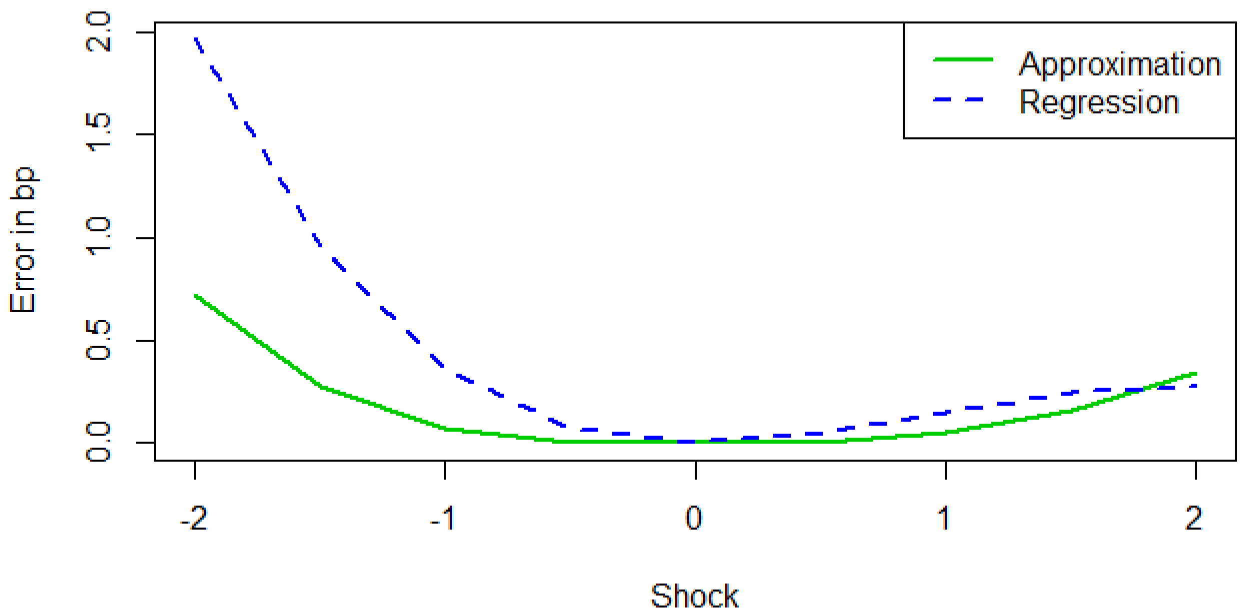


Table 9: $m = 5$ years

Shock	ZC change	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	358,8847	366,5582	362,4199	358,0764	356,7577	356,3085
-3,0	308,3724	313,5776	310,5372	307,8019	307,0901	306,8823
-2,5	257,2668	260,5969	258,4855	256,9026	256,5594	256,4758
-2,0	205,6577	207,6163	206,2650	205,4546	205,3140	205,2866
-1,5	153,6261	154,6357	153,8756	153,5337	153,4892	153,4827
-1,0	101,2452	101,6550	101,3172	101,2159	101,2071	101,2063
-0,5	48,5812	48,6744	48,5899	48,5773	48,5767	48,5767
0,0	-4,3063	-4,3063	-4,3063	-4,3063	-4,3063	-4,3063
0,5	-57,3630	-57,2869	-57,3713	-57,3587	-57,3592	-57,3592
1,0	-110,5403	-110,2675	-110,6054	-110,5040	-110,5128	-110,5120
1,5	-163,7939	-163,2482	-164,0083	-163,6664	-163,7109	-163,7044
2,0	-217,0839	-216,2288	-217,5801	-216,7697	-216,9103	-216,8829
2,5	-270,3742	-269,2094	-271,3208	-269,7380	-270,0812	-269,9977
3,0	-323,6322	-322,1901	-325,2305	-322,4953	-323,2070	-322,9992
3,5	-376,8283	-375,1707	-379,3091	-374,9656	-376,2843	-375,8351

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	7,6735	3,5351	0,8084	2,1270	2,5762
-3,0	5,2051	2,1647	0,5705	1,2823	1,4902
-2,5	3,3301	1,2187	0,3642	0,7075	0,7910
-2,0	1,9586	0,6073	0,2032	0,3438	0,3711
-1,5	1,0096	0,2495	0,0924	0,1369	0,1434
-1,0	0,4098	0,0720	0,0293	0,0381	0,0389
-0,5	0,0932	0,0088	0,0039	0,0044	0,0045
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	0,0761	0,0083	0,0043	0,0038	0,0038
1,0	0,2727	0,0651	0,0362	0,0274	0,0283
1,5	0,5457	0,2144	0,1275	0,0830	0,0895
2,0	0,8551	0,4962	0,3142	0,1736	0,2010
2,5	1,1647	0,9467	0,6362	0,2930	0,3765
3,0	1,4421	1,5984	1,1369	0,4251	0,6329
3,5	1,6576	2,4808	1,8627	0,5440	0,9932

Figure 21: $m = 5$ years



7.3 Tables for the CBB price change approximation

For our implementation of the CBB price change approximation we have taken a face value of 100 $ and a coupon rate of 0.1%. The coupons are paid semiannually starting on the sixth month.

Shock	CBB price	Approx 1	Approx 2	Approx 3	Approx 4	Approx 5
-3,5	36 005,6739	36 776,6602	36 360,6363	35 924,8370	35 792,3995	35 747,2467
-3,0	30 938,1250	31 461,1309	31 155,4807	30 881,0415	30 809,5551	30 788,6645
-2,5	25 810,9799	26 145,6017	25 933,3446	25 774,5256	25 740,0511	25 731,6556
-2,0	20 633,2531	20 830,0724	20 694,2279	20 612,9126	20 598,7918	20 596,0408
-1,5	15 413,0841	15 514,5432	15 438,1307	15 403,8258	15 399,3579	15 398,7050
-1,0	10 157,8239	10 199,0140	10 165,0528	10 154,8884	10 154,0059	10 153,9199
-0,5	4 874,1139	4 883,4847	4 874,9944	4 873,7239	4 873,6687	4 873,6660
0,0	-432,0445	-432,0445	-432,0445	-432,0445	-432,0445	-432,0445
0,5	-5 755,2268	-5 747,5738	-5 756,0640	-5 754,7935	-5 754,8486	-5 754,8460
1,0	-11 090,5294	-11 063,1030	-11 097,0641	-11 086,8997	-11 087,7823	-11 087,6963
1,5	-16 433,5190	-16 378,6322	-16 455,0448	-16 420,7399	-16 425,2078	-16 424,5549
2,0	-21 780,1875	-21 694,1615	-21 830,0060	-21 748,6907	-21 762,8114	-21 760,0604
2,5	-27 126,9112	-27 009,6907	-27 221,9478	-27 063,1288	-27 097,6033	-27 089,2079
3,0	-32 470,4143	-32 325,2199	-32 630,8701	-32 356,4309	-32 427,9173	-32 407,0267
3,5	-37 807,7361	-37 640,7492	-38 056,7730	-37 620,9737	-37 753,4111	-37 708,2583

> > > y2=data.frame("Shock"=shock,error) > names(y2)=c("Shock","Error 1","Error 2","Error 3","Error 4","Error 5") > > y2=xtable(y2) > digits(y2)=xdigits(y2) > print(y2,format.args = list(big.mark = " ", decimal.mark = ","),include.rownames = FALSE)

Shock	Error 1	Error 2	Error 3	Error 4	Error 5
-3,5	770,9862	354,9624	80,8370	213,2744	258,4273
-3,0	523,0059	217,3557	57,0835	128,5700	149,4606
-2,5	334,6218	122,3647	36,4543	70,9288	79,3243
-2,0	196,8193	60,9748	20,3406	34,4613	37,2124
-1,5	101,4591	25,0466	9,2584	13,7263	14,3791
-1,0	41,1900	7,2289	2,9355	3,8180	3,9040
-0,5	9,3709	0,8806	0,3900	0,4451	0,4478
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,5	7,6531	0,8372	0,4333	0,3782	0,3808
1,0	27,4264	6,5347	3,6297	2,7471	2,8331
1,5	54,8867	21,5258	12,7791	8,3112	8,9640
2,0	86,0260	49,8185	31,4968	17,3760	20,1270
2,5	117,2204	95,0366	63,7824	29,3078	37,7033
3,0	145,1943	160,4559	113,9834	42,4969	63,3876
3,5	166,9869	249,0370	186,7623	54,3249	99,4778