Optimal Asset-Liability Management of Issuers of Variable Annuities with Guarantees

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Abstract

This article considers the optimal fund allocation for an insurance company that issues variable annuities with riders such as GMWBs. We analyze the problem from the point of view of the managers of the insurance company. We optimize with respect to two criteria: The utility of the profitability of the scheme, and also the level of asset-liability concordance. For the asset-liability management part of the criterion, we rely on a Redington type immunization where the duration of assets and liabilities should be matched and the convexity of the assets should be as high as possible in comparison to the convexity of the liabilities. The asset portfolio consists of stocks, bonds, and cash, and are modeled using diffusion process. The problem is solved using Hamilton-Jacobi-Bellman equations. Illustrations and discussions are provided.

Keywords

Variable Annuity. GMWB. Multivariate separable utility function. Hamilton-Jacobi-Bellman. Duration. Convexity. Redington Immunization.

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Introduction

In North America, Equity Indexed Annuities (EIAs) and Variable Annuities (VAs) constitute an important market. EIAs are linked to an equity index, typically the S&P 500 index or an international index, because they are considered to be fixed annuities by insurance regulators, they fall under protection of state's guaranty funds. VAs are linked to equity securities chosen by the investor. VAs were created in the US in the 1950s and became widespread in Japan in the 1990s; they also have become popular in Europe at the end of the 2000s. In Canada, segregated fund contracts and guaranteed investment funds are popular. Segregated funds also consist of a pool of investments in securities but they are fully segregated from the company's general investment funds. In the UK, the most common instruments are Compulsory Purchase Annuities, such as conventional annuities, with-profit annuities and unit-linked annuities. A Unit Linked Insurance Plan is a product launched in India where policyholders have the option to select the type of funds they prefer. In France and in continental Europe also, similar products are increasingly made available to the public.

In the actuarial literature, relatively many papers take the retiree's standpoint rather than the insurance company's side. The retiree has the option to annuitize his plan balance, so to select an asset mix upon which the annuity payments will be based. This asset mix is one possible focus of research. Charupat and Milevsky (2002) derived the optimal asset allocation for a VA contract in the payout phase, where the retiree's expected utility of consumption is maximized under the condition that the account balance is annuitized at fixed retirement dates. Unlike Charupat and Milevsky (2002), Milevsky and Young (2007) focuses on the optimal annuitization timing and asset mix strategies of a utility-maximizing retiree endowed with a fixed annuity. Milevsky and Kyrychenko (2007) detected the change on asset mix and found that policyholders are holding more equity when guarantees are offered. Most of the above papers used a constant relative risk aversion(CRRA) or constant absolute risk aversion(CARA) utility function, while Huang and Milevsky (2008) considered the general hyperbolic absolute risk aversion(HARA) case.

Another research strand focuses on the annuities when the retirement portfolio includes other products. Chen and Milevsky (2003) computed the optimal mix between traditional assets and immediate annuities using the scenario studies and obtained the sensitivities of weights with respect to risk aversion, age, and other factors. Xiong, Idzorek, and Chen (2010) considered a similar problem with deferred annuities. Kartashov, Maurer, Mitchell, and Rogalla (2013) assessed the impact of allocation of variable annuities on household welfare. The benefits of the variable annuities depend on the performance of the underlying portfolio under different stochastic and systematic mortality scenarios. Both Blanchett (2012) and Horneff, Maurer, Mitchell, and Rogalla (2013) specified the allocation problem for the popular Guaranteed Minimum Withdraw Benefit (GMWB) product. Ostaszewski (2013) considered an empirical problem of allocation of an individual investment portfolio in the context of a guarantees provided by income from a social insurance scheme.

Next, we examine several papers that look at the insurance company's side. For insurance companies that issue variable annuities, two core issues are the pricing of contracts and the hedging of risks. A large quantity of work has been done for the former point that we omit here. One can refer to the introduction of

Kelani and Quittard-Pinon (2014) for several papers concerning hedging in life insurance and variable annuities. Let us mention here several additional papers. Coleman, Li, and Patron (2006) compared the effectiveness of hedging against equity and interest rate risk, whether using underlying assets or liquid options. Cox and Lin (2007) focused on the hedging of life annuities and annuities against mortality risk.

Most of the papers from the insurer's side take the directions mentioned above, which are not pursued in this paper. The following papers are also from the insurer's side, but instead of a risk minimization that is usually used for hedging, they use a utility framework. Devolder, Princep, and Fabian (2003) consider a defined contribution pension plan whose benefits are paid with an annuity. Their paper neglects mortality risk when choosing the best investment strategies for managers and finds explicit solutions of asset allocation before and after retirement for the CRRA and CARA utility functions.

Let us now examine how financial risk is modeled in the related actuarial literature. The common interest rate models include the constant model used in Charupat and Milevsky (2002) and Milevsky and Young (2007) and the Vasicek model as used in Coleman, Li, and Patron (2006). The models used to simulate risky assets include the Geometric Brownian Motion (GBM) model used in Charupat and Milevsky (2002) and Milevsky and Young (2007), the GBM model with dividends used in Coleman, Li, and Patron (2006), the Merton's Jump Diffusion model used in Coleman, Li, and Patron (2006) and the Lévy process approach used in Kelani and Quittard-Pinon (2014).

In addition to the above two kinds of assets, models comprising three or even four kinds of assets also exist. A bond account is the most commonly additional asset. The common bond models include the stochastic diffusion model used in Boulier, Huang, and Taillard (2001) and Battocchio and Menoncin (2004) and the yield smoothing approach used in Binsbergen and Brandt (2007).

Annuities can be modeled in several ways, depending on interest rates as in Milevsky and Young (2007), or depending on assumed interest rates as in Charupat and Milevsky (2002) and Kartashov, Maurer, Mitchell, and Rogalla (2013), or using the discrete model as in Cox and Lin (2007) and Horneff, Maurer, Mitchell, and Rogalla (2013).

Then, we examine the mortality risk models used in the above mentioned articles. The market is assumed complete under mortality risk if the number of policyholders is large enough, as in Coleman, Li, and Patron (2006) and Devolder, Princep, and Fabian (2003). Mortality can be regarded as part of the extra fees with other expense charges, as in Milevsky and Kyrychenko (2007). Other approaches include the exponential model used in Charupat and Milevsky (2002) and the Gompertz-Makeham model used in Charupat and Milevsky (2002) and Milevsky and Young (2007).

While the key optimization problem is about maximization of the wealth of the owner of the insurance firm offering the variable product, this optimization is under specified asset-liability management constraints. Our model, therefore, in contrast to existing literature, captures what is common practice of the insurance industry in variable annuities: A trade off between maximizing company wealth, and tools used for management of asset-liability risks, within the framework of providing guarantees to the customer.

1 Actuarial and Financial Setting

We construct a fund that represents the wealth of an insurance company. The analysis can be separated into two time periods: before and after retirement. Before retirement, we have an optimal portfolio problem where the premiums paid by the policyholders make up the insurer's initial wealth and where the objective is to find the optimal allocation of assets maximizing the expected utility of wealth until retirement. Some changes can be made if we want to model the characteristics of certain riders, e.g., GMWB, where withdrawals are allowed before retirement. At retirement, part or all of the fund is used to purchase an annuity. The benefits can be calculated using the annuity models such as those described above. After retirement, the optimization problem for insurers can be regarded as finding allocation strategies to optimize the remaining funds at policyholders' limit ages.

We consider an insurance company that issues variable annuities with guarantees, specifically guaranteed minimum withdrawal benefits(GMWBs). The problem is to find the "best" investment strategy for the assets that back the pension liabilities during the existing period of the annuities. In this section, we first examine the mechanism of a GMWB. Then we introduce the market structure under which the optimal asset allocation problem is defined.

1.1 GMWBs

Originally, the payment from a variable annuity used an Assumed Interest Rate (AIR), where the payment changes based on the actual return of the underlying investments compared to the AIR. For example, if the AIR is 4%, as long as the underlying fund earns more than 4%, say 10%, payments will increase by a factor $\frac{1.1}{1.04} \approx 1.06$. If the underlying fund earns less than 4%, say -10%, payments will decrease by a factor $\frac{0.9}{1.04} \approx 0.87$. Because payments depend on the performance of the underlying investments, the variable annuity is risky. Its risk can be managed with the creation of a GMWB.

A GMWB is a combination of an insurance and an investment. There are two types of GMWBs: finite-life GMWBs and lifetime GMWBs. A finite-life GMWB is a contract where the holder can withdraw guaranteed periodic amounts up to the initial capital. The GMWB terminates once the initial investment has been withdrawn; any remaining funds are returned to the policyholder at maturity. For example, if one invests 100 euros in a GMWB that guarantees a 5% annual payment, then the maturity of the contract is set at 20 years.

We now describe in detail the mechanism of a lifetime GMWB, which is also called a GLWB, where L stands for lifetime. With a typical lifetime GMWB, the policyholder deposits a lump sum in a pre-retirement year, say at age $T_0 = 50$, to buy the variable annuity with GMWB rider. We denote C_0 this lump sum, which includes the premium for the VA and the price of the GMWB option. Then, the policyholder has the right to choose the composition of the assets in his/her sub account. Let us assume that 70% of the initial contribution is virtually invested in risky assets and 30% in bonds. The management fee for the stock and bond accounts is 0. We further assume that once the contribution is invested, it follows a buy-and-hold strategy and no further contribution is made during the accumulation period. Once the variable annuity is bought, the benefit base account and contract value account are set up. While the contract value account is the actual value of the investments in the annuity's sub accounts, the benefit base is what future guaranteed income payments will be based on. Typically, the benefit base initially equals C_0 and automatically increases by a certain percentage each year for a specified time period. If the contract value account is greater than the benefit base at the end of a certain year, the benefit base account will be periodically set equal to the higher contract value account. The benefit base account remains unchanged after retirement.

For simplicity, we assume that at the age of retirement, normally $T_R = 65$, the policyholder decides to withdraw a lump sum equal to the contract value or to receive an annuity, so to withdraw monthly or annual payments from the next period on. Once the withdrawals begin, the benefit base no longer increases automatically.

The guaranteed minimum payments are distributed as a percentage of the benefit base until death. We postulate that the policyholder can withdraw up to 5% of the investment amount each year for life, no matter how long he/she lives or how bad his/her investments perform. The policyholder has the option to choose his/her withdrawal strategy, but we postulate that he/she will only withdraw 5% per year. At the death time of the policyholder, if a positive contract value still exists, it is returned to the beneficiary. If the policyholder dies before retirement, his/her contract value is also returned to the beneficiary at his/her supposed retirement age.

1.2 Financial Market

The insurance company holding the premiums invests them in cash, stocks and bonds. The portfolio holds a quantity $\theta_0(t)$ of cash, and numbers $\theta_S(t)$ of risky assets and $\theta_B(t)$ of bonds.

In Table 1, the left-hand-side column lists the company's assets: the equity investment, the real estate investments, B_K, the position in bonds and S^0, the cash. The right-hand-side column displays the capital and liability of the insurance company.

Table 1: Overall Framework/space

$\left.\begin{array}{c}\text{stocks}\\ \text{real estates}\end{array}\right\}: S^1$	capital
bonds: B_K	
cash: S^0	liabilities: VA and guarantees

Default risk is neglected in this study. We assume that the instantaneous credit-risk-free interest rates follows a Vasicek model:

$$dr(t) = a(b - r(t))dt + \sigma_r dW_r(t), \quad r(0) = r_0, \tag{1}$$

where W_r is a Brownian motion in the real world. The explicit solution to (1) is

$$r(t) = b + (r_0 - b)e^{-at} + \sigma_r \int_0^t e^{-a(t-s)} dW_r(s),$$

and we have for the corresponding cash asset S^0:

$$S^0(t) = e^{\int_0^t r(s)ds}.$$

The expectation of this latter quantity equals

$$e^{m_t + \frac{1}{2}\Sigma_t^2},$$

where

$$m_t = bt + (r_0 - b)\frac{1 - e^{-at}}{a}, \quad \Sigma_t^2 = \sigma_r^2 \int_0^t (\frac{1 - e^{-a(t-u)}}{a})^2 du.$$

For the rolled-over position in bonds, we use the diffusion equation introduced in Boulier, Huang, and Taillard (2001). The strategy for constructing a rolling bond is achieved by continuously rebalancing a portfolio of zero-coupon bonds and allow us to reproduce bonds with a single time to maturity K. For such a bond, the price follows the equation

$$\frac{dB_K(t)}{B_K(t)} = r(t)dt - \sigma_K(dW_r(t) + \lambda_r dt), \tag{2}$$

where λ_r is a constant risk premium and

$$\sigma_K = \frac{1 - e^{-aK}}{a}\sigma_r.$$

As mentioned before, the insurance company invests part of its funds in risky assets. This investment in risky assets of the insurance company includes a combination of stocks and real estate participations. We assume that the total dynamics of this combination is represented by S^1. We call S^1 stock for simplicity from now on. The policyholder has the right to choose a portfolio, on which his future benefits will be based. He/she also chooses a combination of stocks and real estates to represent the risky investment in his/her underlying. We assume that the dynamics of this combination is represented by S^2. Note that S^2 only acts as an index on which the policyholder's benefits will be based; it is not possessed by the insurer or the policyholders. The real-world dynamics of S^1 and S^2 are given below:

$$\begin{cases} \frac{dS^1(t)}{S^1(t)} = r(t)dt + \sigma_1(dW_1(t) + \lambda_1 dt) - \delta_1(t)dt, \\ \frac{dS^2(t)}{S^2(t)} = r(t)dt + \sigma_2(dW_2(t) + \lambda_2 dt) - \delta_2(t)dt, \\ \qquad < dW_1, dW_2 >= \rho_{12}dt. \end{cases} \tag{3}$$

W_1, W_2 are correlated Brownian motions and λ_1, λ_2 are constant premiums. Each stock pays dividends continuously at a rate $\delta_{i=1,2}(t)$ per unit of time. S^1 represents the combination in which the company really invests while S^2 represents the combination used for the computation of that the policyholder's subaccount appreciation. If the stock market is assumed to be correlated with the bond market, then $< dW_1, dW_r > = \rho_{r1}dt$, $< dW_2, dW_r > = \rho_{r2}dt$. So, the variance-covariance matrix of B_K, S^1, S^2 is

$$MVC = \begin{pmatrix} \sigma_K^2 & -\sigma_K\sigma_1\rho_{r1} & -\sigma_K\sigma_2\rho_{r2} \\ -\sigma_K\sigma_1\rho_{r1} & \sigma_1^2 & \sigma_1\sigma_2\rho_{12} \\ -\sigma_K\sigma_2\rho_{r2} & \sigma_1\sigma_2\rho_{12} & \sigma_2^2 \end{pmatrix} \tag{4}$$

We can construct uncorrelated Brownian motions $\widetilde{W_r}$, $\widetilde{W_1}$ and $\widetilde{W_2}$, where

$$\begin{cases} d\widetilde{W_r} = dW_r - \frac{\sigma_K}{2}dt, \\ d\widetilde{W_1} = \frac{1}{\sqrt{1-\rho_{r1}^2}}\left(dW_1 - \rho_{r1}dW_r + \left(\frac{\sigma_1}{2} + \frac{\rho_{r1}\sigma_K}{2}\right)dt\right), \\ d\widetilde{W_2} = \frac{\sqrt{1-\rho_{r1}^2}}{\sqrt{(1-\rho_{r1}^2)(1-\rho_{r2}^2)-(\rho-\rho_{r1}\rho_{r2})^2}}\Bigg(dW_2 - \frac{\rho-\rho_{r1}\rho_{r2}}{1-\rho_{r1}^2}dW_1 \\ \qquad - \left(\frac{\rho_{r2}-\rho_{r1}\rho}{1-\rho_{r1}^2}\right)dW_r + \left(\frac{\sigma_2+\rho_{r2}\sigma_K-\sigma_2\rho_{r1}^2-\rho\sigma_1-\rho\rho_{r1}\sigma_K+\rho_{r1}\rho_{r2}\sigma_1}{2(1-\rho_{r1}^2)}\right)dt\Bigg) \end{cases} \tag{5}$$

The real-world dynamics of the bond and the risky investments become

$$\begin{cases} \frac{dB_K(t)}{B_K(t)} = (r(t) - \sigma_K\lambda_r - \frac{\sigma_K^2}{2})dt - \sigma_K d\widetilde{W_r}, \\ \frac{dS^1(t)}{S^1(t)} = (r(t) + \sigma_1\lambda_1 - \delta_1 - \frac{\sigma_1^2}{2})dt + \sigma_1\rho_{r1}d\widetilde{W_r} + \sigma_1\sqrt{1-\rho_{r1}^2}d\widetilde{W_1}, \\ \frac{dS^2(t)}{S^2(t)} = (r(t) + \sigma_2\lambda_2 - \delta_2 - \frac{\sigma_2^2}{2})dt + \sigma_2\rho_{r2}d\widetilde{W_r} + \frac{\sigma_2(\rho-\rho_{r1}\rho_{r2})}{\sqrt{1-\rho_{r1}^2}}d\widetilde{W_1} \\ \qquad + \sigma_2\sqrt{\frac{(1-\rho_{r1}^2)(1-\rho_{r2}^2)-(\rho-\rho_{r1}\rho_{r2})^2}{1-\rho_{r1}^2}}d\widetilde{W_2}, \end{cases} \tag{6}$$

For simplicity, we omit the $\widetilde{\cdot}$ and rely on uncorrelated Brownian motions until the end of the paper.

1.3 Company Assets

As already stated, the insurance company invests $\theta_0(t)$ in cash, $\theta_S(t)$ in risky assets and $\theta_B(t)$ in bonds, i.e.,

$$F_t = \theta_0(t)S^0(t) + \theta_S(t)S^1(t) + \theta_B(t)B_K(t).$$

The associated SDE is therefore

$$\begin{aligned} dF_t &= \theta_0(t)dS^0(t) + \theta_S(t)dS^1(t) + \theta_B(t)dB_K(t) \\ &\quad + d\theta_0(t)(S^0(t) + dS^0(t)) + d\theta_S(t)(S^1(t) + dS^1(t)) + d\theta_B(t)(B_K(t) + dB_K(t)). \end{aligned}$$

Before time T, no withdrawals exist and the company is self-financed. According to the self-financing condition,

$$dF_t = \theta_0(t)dS^0(t) + \theta_S(t)dS^1(t) + \theta_B(t)dB_K(t).$$

Using Eqs. (6), we have:

$$\begin{aligned} dF_t &= [F_t r(t) + \omega_1(t)\lambda_1\sigma_1 - \omega_2(t)\lambda_r\sigma_K - \omega_1(t)\delta_1(t)]\,dt + \\ &\quad \omega_1(t)\sigma_1\sqrt{1-\rho_{r1}^2}dW_1(t) + (\omega_1(t)\sigma_1\rho_{r1} - \omega_2(t)\sigma_K)dW_r(t), \end{aligned}$$

where

$$\omega_1(t) = \theta_S(t)S^1(t), \quad \omega_2(t) = \theta_B(t)B_K(t), \quad \omega_1(t) + \omega_2(t) = F_t - \theta_0(t)S^0(t).$$

In vector form,

$$dF_t = [F_t r(t) + \omega(t)'(\sigma\lambda - \delta)]dt + \omega(t)'\sigma dW_t, \tag{7}$$

where

$$\omega(t) = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix},\ \lambda = \begin{pmatrix} \frac{\lambda_1 - \rho_{r1}\lambda_r}{\sqrt{1-\rho_{r1}^2}} \\ \lambda_r \end{pmatrix},\ \sigma = \begin{pmatrix} \sigma_1\sqrt{1-\rho_{r1}^2} & \sigma_1\rho_{r1} \\ 0 & -\sigma_K \end{pmatrix},\ \delta = \begin{pmatrix} \delta_1 \\ 0 \end{pmatrix},\ dW_t = \begin{pmatrix} dW_1(t) \\ dW_r(t) \end{pmatrix}.$$

1.4 Contracts

Next, we model the variable annuity and the GMWB guarantee. We denote $t_i, i = 0, 1, 2\cdots$ a sequence of constant times. The time interval is determined by the contract, i.e., if the contract payments are annual, the time interval is one year, if the contract payments are monthly, the time interval is one month, and so on. Further, $t_0 = 0$.

We use β to denote the benefit base and C to represent the contract value. C_{t_i} is the contract value after withdrawals. $\beta_0 = C_0$ is the original lump sum used for the purchase of the contract. For simplicity, we construct an index P that tracks the underlying fund value. This index does not represent the assets of the insurer; it is the index on which the policyholder's contract is based and is made of S^2 and B_K. Indeed,

$$\frac{P(t)}{P(0)} = \nu_1 \frac{S^2(t)}{S^2(0)} + \nu_2 \frac{B_K(t)}{B_K(0)}, \tag{8}$$

where $\nu_1 = 0.7$, $\nu_2 = 0.3$.

If the policyholder chooses to receive an annuity, he/she starts receiving monthly or annual level payments from time $t_i > T \triangleq T_R - T_0 = 15$, where, as mentioned in subsection 1.1, T_R is the retirement age and T_0 is the age at which the contract begins. Here, we assume that the constant annual withdrawal rate is 5%. For $t_i > T$, the investor is guaranteed to be able to make a withdrawal of

$$w_{t_i} = \begin{cases} 0 & \text{if } t_i \leqslant T, \\ 5\%\beta_{t_{i-1}} & \text{if } t_i > T. \end{cases} \tag{9}$$

at each date t_i.

We further define r_g as the percentage increase of the benefit base. According to the step up feature, before T, the benefit base β_{t_i} is set to the higher of $\beta_{t_{i-1}}(1 + r_g)$ and of the contract value after withdrawals, i.e., C_{t_i}. After T, the benefit base β_{t_i} remains unchanged, such that

$$\beta_{t_i} = \begin{cases} \max(\beta_{t_{i-1}}(1 + r_g), C_{t_i}) & \text{if } t_i \leqslant T, \\ \beta_T & \text{if } t_i > T. \end{cases} \tag{10}$$

The contract value changes with the value of underlying assets. Net of withdrawals, it never falls below zero, by construction:

$$C_{t_i} \quad = \quad (C_{t_{i-1}} \cdot \frac{P_{t_i}}{P_{t_{i-1}}} - w_{t_i})^+, \tag{11}$$

Based on the above equations, the proceeds for the original VA plus GMWB for the policyholder at time $t_i = T$ is either a lump sum equal to the contract value or a sequence of withdrawals, i.e.,

$$\text{Payoff}(T) = \max(5\%\beta_T \ddot{a}_R(T), C_T).$$

where $5\%\beta_T \ddot{a}_R(T)$ is the market value of lifelong withdrawals and $\ddot{a}_R(T)$ is the market price at time T of an immediate annuity with payments of one euro per annum beginning at time T until death.

We can derive a different interpretation. GMWB riders represent embedded financial put options on the contract values. Milevsky and Salisbury (2006), which is the initial paper on pricing and hedging finite-life GMWB products, decompose the contract into a term certain component and a quanto Asian put option. Inspired by them, Piscopo (2009) decomposes the lifetime GMWB contract into a life annuity and a portfolio of quanto Asian put options.

1.5 Mortality Model

We use the Gompertz-Makeham mortalitymodel. In this model, the force of mortality ψ can be written as:

$$\psi(t) = \chi + \frac{1}{b} e^{\frac{t-m}{b}}, \tag{12}$$

where χ is a positive constant measuring accidental deaths linked to non-age factors, while m and b are modal and scaling parameters of the distribution, respectively. The corresponding survival probability is

$$_t p_{t_0} = e^{-\chi(t-t_0)+e^{\frac{t_0-m}{b}\left(1-e^{\frac{t-t_0}{b}}\right)}}.$$

In our framework we assume that τ is independent of all the other stochastic variables affecting the financial market.

2 Optimization Problem

2.1 Asset-liability Matching Indicators

Asset-liability mismatches are an important concern for insurance companies and pension funds. Several indicators exist that allow us to test the different aspects of asset-liability matching levels.

One can consider for instance the difference between the total risk of assets and that of liabilities. The tracking error, defined as the standard deviation of the difference in the returns of assets and liabilities, measures this difference. In symbolic form,

$$\text{tracking error} = \mathrm{SD}[R_A - R_L],$$

where R_A and R_L are the returns of assets and liabilities, respectively. SD is the standard deviation. It is also possible to construct indexes from different asset-liability matching strategies, like dedication and immunization strategies.

In a dedication strategy, assets and liabilities provide the same total cash flows in each period. From this strategy, we can construct the index maturity gap. The maturity gap is defined as follows: assets and liabilities are grouped into time buckets according to maturity or the time until the first possible resetting date of interest rates. For each time bucket,

$$\text{maturity gap} = M_A - M_L,$$

where M_A and M_L are maturities for assets and liabilities. When the maturity gap is zero, the company is protected against interest rate moves.

A commonly used immunization strategy is duration matching. Under this strategy, we construct a duration matching level index that is defined as the difference between the durations of assets and liabilities:

$$\text{duration matching level} = D_A - D_L.$$

The duration gap is also used to measure the risks related to interest rate changes. It is defined as

$$\text{duration gap} = D_A - D_L \frac{\text{assets}}{\text{liabilities}}.$$

When the duration gap is positive, the company suffers from an increase of interest rates; when the duration gap is negative, the company suffers from a decrease of interest rates. When the duration gap is zero, the company is immunized against interest rate risk. The duration gap improves the maturity gap by taking into account the timing and market value of cash flows rather than the horizon maturity.

Utility functions are increasing functions, so the immunization target can be expressed as

$$\frac{1}{|D_A - D_L|},$$

or as

$$\frac{1}{|D_A - D_L \frac{\text{assets}}{\text{liabilities}}|}.$$

The Redington immunization strategy, named after the British actuary Frank Redington, imposes an additional condition on convexity beyond duration and asset and liability value matching. See Redington (1952). Specifically, this strategy requires that the convexity of assets be greater than that of liabilities. However, it is not always easy to enforce this condition in practice. Rather, we impose a related condition: we maximize the ratio of the convexity of assets to the convexity of liabilities. Then, we construct an index that consists of two parts:

$$\xi \frac{1}{|D_A - D_L|} + \zeta \frac{C_A}{C_L}, \tag{13}$$

where D denotes duration, C denotes convexity, α and ζ are given coefficients: We use this new indication in the remainder of the test as an asset- liability management criterion. It should be noted that using this criterion results in a possible tradeoff between duration match and convexity gain. While such tradeoff has been largely ignored in existing literature, it is a real issue in the practical management of insurance firms, as well as the tradeoff between convexity and yield, which has received much more attention in the existing literature.

The liquidity mismatch index (LMI) defined by Brunnermeier, Markus, Gorton, and Krishnamurthy (2011) and Brunnermeier, Markus, Gorton, and Krishnamurthy (2013) is also worth mentioning. This index measures the mismatch between the market liquidity of assets and the funding liquidity of liabilities. The market liquidity of assets is the ability to quickly convert them into cash. The funding liquidity of liabilities is the ability to raise money by borrowing using the assets as collateral. The LMI is defined as the difference between the liquidities of assets and liabilities, and weighs each asset and liability by a liquidity weight η. In symbolic form,

$$LMI_t = \sum_k \eta_{t,A_k} x_{t,A_k} - \sum_{k'} \eta_{t,A_{k'}} x_{t,A_{k'}}.$$

The assets x_{t,A_k} and liabilities $x_{t,A_{k'}}$ vary with time and with the class k or k' of the asset or liability considered. η_{t,A_k} and $\eta_{t,A_{k'}}$ are liquidity weights. Beyond the LMI, the liquidity coverage ratio (LCR) of Basel 3 and the liquidity creation measure introduced by Berger and Bouwman (2009) are other relevant indicators.

2.2 Objective Function

We assume that the insurance company considers two elements in its optimization strategy: the expected value of the fund at the contract maturity and/or the company's asset liability matching criteria. Its managers are concerned about both profitability and risk management.

First, we consider optimization of the utility of wealth. There are two situations for the contract maturity. In the first situation, the policyholder prefers to receive a lump sum equal to the balance in his/her sub-account at the retirement age. The decision is made at the retirement age. Then, the contract maturity is the smaller of the retirement age T and death age τ. In this case, the optimization program is

$$\omega(t) = \text{argmax } \mathbb{E}_P \left[U_1(F_\tau) e^{-\kappa\tau} \mathbb{1}_{\tau \leqslant T} + U_1(F_T) e^{-\kappa T} \mathbb{1}_{\tau > T} \right], \tag{14}$$

where κ is the subjective discounting factor that represents the degree of an insurer's patience, U_1 is the utility function and F is subject to Eq. (7).

In the second situation, the policyholder prefers to receive an annuity. The decision is also made at the retirement age. Then, the contract maturity is the client's death time τ. It is necessary to make some changes to the original optimization problem and its constraints. The portfolio is not self-financed any more. The changes in the portfolio are made of discrete withdrawals. We can consider them as jumps.

On $t \in [0,\tau] \backslash \{t_i\}$, the constraint condition of F_t is still given by

$$dF_t = \theta_0(t) dS^0(t) + \theta_S(t) dS^1(t) + \theta_B(t) dB_K(t),$$

for $t_i = T+1, \cdots, \llcorner\tau\lrcorner$. $\llcorner\tau\lrcorner$ means the largest integer smaller than τ. At times $t_i = T+1, \cdots, \llcorner\tau\lrcorner$, we have the following jump condition:

$$F(t_i) = F(t_i^-) - w_{t_i},$$

which means the company pays w_{t_i} to its policyholders at time t_i.

Using the notation introduced above, Eq. (7) can be modified as

$$\begin{cases} dF_t = [F_t r(t) + \omega(t)'(\sigma\lambda - \delta)]dt + \omega(t)'\sigma dW_t, , & \text{For } t \in [0,\tau]\backslash\{T+1, \cdots, \llcorner\tau\lrcorner\} \\ F(t_i) = F(t_i^-) - w_{t_i}, & \text{For } t_i = T+1, \cdots, \llcorner\tau\lrcorner \end{cases} \tag{15}$$

The optimization problem can be modified as

$$\omega(t) = \text{argmax } \mathbb{E}_P \left[U_1(F_\tau) e^{-\kappa\tau} \right], \tag{16}$$

where U_1 is the utility function and F is subject to Eq. (15).

As we mentioned before, the company not only considers the optimization of utility of wealth, but also cares about asset liability matching. For the latter point, we can refer to the last subsection for a survey about asset liability matching criteria. In fact, we optimize the problem in two steps, the details of which will be specified in the next section.

Here, we use the duration matching index and synthesize the above two points to the following objective function:

$$\omega(t) = \text{argmax} \left[\alpha \mathbb{E}_P \left[U_1(F_\tau) e^{-\kappa\tau} \right] + (1-\alpha) U_2 \left(\frac{1}{|D_A - D_L|} \right) \right], \tag{17}$$

where U_2 is the utility function, D_A and D_L are the durations of assets and liabilities respectively, and where we have assumed that the total utility function is separable.

3 Results

3.1 Solution to the first wealth optimization problem

When dealing with problem (14), we can write the maximization problem as follows:

$$\omega(t) = \text{argmax } \mathbb{E}_0^P \left[\int_0^T U_1(F_s) \psi(s) e^{-\int_0^s \kappa + \psi(u) du} ds + U_1(F_T) e^{-\int_0^T \kappa + \psi(u) du} \right]$$

where ψ is the force of mortality and is independent of financial risk. The factors that multiply the utility terms are respectively a death density and a death survival probability, times a discount factor in κ.

Assume that the utility function U_1 is CRRA with risk aversion coefficient γ, i.e., $U_1(c) = \frac{c^{1-\gamma}}{1-\gamma}$. According to Proposition 4.1, the solution is

$$\begin{cases} \frac{\omega_1^*(t)}{F} = \frac{\sigma_1 \lambda_1 - \delta_1}{\gamma}, \\ \frac{\omega_2^*(t)}{F} = \frac{\sigma_r (1-\gamma)}{\sigma_K a \gamma} (e^{a(t-T)} - 1) - \frac{\sigma_K \lambda_r}{\gamma}. \end{cases} \tag{18}$$

3.2 Solution to the second wealth optimization problem

When dealing with problem (16), we can write the maximization problem as follows:

$$\omega(t) = \text{argmax } \mathbb{E}_t^P \left[\int_0^\infty \psi(s) U_1 \left(F(s) \right) e^{-\int_0^s \kappa + \psi(u) du} ds \right].$$

We also assume that U_1 is CRRA. According to Proposition 4.2, the solution is

$$\omega^*(t) = (\sigma\sigma')^{-1} (F - \sum_{i: t_i > t} w_{t_i} e^{-\int_t^{t_i} r(s) ds}) \frac{\sigma (0\ \sigma_r)' A_2 + (\sigma\lambda - \delta)}{\gamma}. \tag{19}$$

3.3 Solution to the risk management problem

Next, we consider the durations of assets and liabilities. For the liabilities, as withdrawals are discrete, we can use the following formula to calculate the Macaulay duration

$$\text{Duration of liabilities}(t) = \frac{\Sigma_{t_i=T+1}^{\llcorner\tau\lrcorner} t_i w_{t_i} e^{-\int_t^{t_i} r(s)ds}}{\Sigma_{t_i=T+1}^{\llcorner\tau\lrcorner} w_{t_i} e^{-\int_t^{t_i} r(s)ds}}. \tag{20}$$

For the assets, companies are holding cash, stocks and bonds. We first calculate the duration of each part and use

$$\text{Duration of assets} = (1 - \frac{\omega_1}{F} - \frac{\omega_2}{F}) \cdot D(\text{cash}) + \frac{\omega_1}{F} D(S^1) + \frac{\omega_2}{F} D(B_K) \tag{21}$$

to calculate the asset duration. The duration of cash can be regarded as zero. According to Farrell Jr. (1985), the duration of stocks can be approximated by $\frac{1}{\delta_1}$. This is, of course, only a "first order" approximation, because in practice duration of stocks is strongly affected by the ability of managers to respond to interest rate changes by adjusting the projects the firm invests it, and generally, is shorter than the "first order" approximation produced by the reciprocal of the dividend yield. The Macaulay Duration of a zero coupon bond equals its term to maturity K.

Therefore, the optimization problem becomes

$$\omega^*(t) = \text{argmax}\left[U_2\left(\frac{1}{|\frac{\omega_1}{F\ \delta_1} + \frac{K\omega_2}{F} - D_L|}\right)\right]. \tag{22}$$

Dealing with the above problem alone gives out infinite optimal solutions, which is making the denominator equal to 0,

$$\frac{\omega_1}{F\ \delta_1} + \frac{K\omega_2}{F} = D_L.$$

As mentioned before, we will deal with this problem in two steps. The first step concerns the duration matching part. The second step concerns the convexity improvements. We add an additional constraint that the company should hold a minimum level of cash to makes sure its solvency. We suppose that this minimum level is 5% of the company's fund, that is $\omega_1 + \omega_2 = 0.95F$. The optimization algorithm that we are actually processing is:

$$\begin{cases} \frac{\omega_1}{F\ \delta_1} + \frac{K\omega_2}{F} = D_L, \\ \omega_1 + \omega_2 = 0.95F. \end{cases}$$

Solving the above system of equations, we get the results

$$\begin{cases} \omega_1 = \frac{F\delta_1 D_L - 0.95KF\delta_1}{1-F\delta_1}, \\ \omega_2 = \frac{0.95F - F\delta_1 D_L}{1-F\delta_1}. \end{cases}$$

3.4 Numerical simulation details

The calculations of D_A Eq. (21) analytical, while the calculations of D_L in Eq. (20) depend on the trajectory of Monte-Carlo method. In this part, we display the numerical details for calculating D_L.

First, generate Nb branches of $(r_t, B_K(t), S_1(t), S_2(t))$. Then, in each scenario calculate the benefit base account and contract value to obtain the cash flows of liabilities w_t. Use Eqs. (20) to obtain the duration and convexity of liabilities in each scenario and average all these durations to get the desired D_L.

4 Illustration

In this section, we set out a numerical illustration in order to analyze the dynamic behavior of the optimal portfolio strategy derived in the above subsection. The parameters of the financial market and the mortality model are listed in Table 2.

Table 2: Parameters for illustration

Interest rate		**Risky investment S^1**	
mean reversion, a	0.2	Risk premium, λ_1	0.3
mean rate, b	0.05	stock own volatility, σ_1	0.2
volatility, σ_r	0.02	dividends, δ_1	0.02
initial rate, r_0	0.03		
		Risky investment S^2	
Fix-maturity bond		Risk premium, λ_2	0.2
maturity, K	20	stock own volatility, σ_2	0.1
market price of risk, λ_r	-0.15	dividends, δ_2	0
Gompertz-Makeham model		correlation between stocks, ρ_{12}	0.2
non-age factor, χ	0.01	correlation between S^1 and B_K, ρ_{r1}	0.06
modal, m	92.63	correlation between S^2 and B_K, ρ_{r2}	0.06
scaling, b	8.78		

Figure 1 shows the results of Eq. (23) with respect to the relative risk aversion level. It is not surprising to find that the higher the relative risk aversion level, the lower the risky asset optimal share.

As the results to problems (16) and (17) are not analytical, we use Monte-Carlo method to obtain the average behavior. However, even with $N = 100,000$ scenarios, the results are volatile, so we calculate the mean excluding the highest and lowest k data values where $k = N$ percent/2. Figures 2 and 3 show the results of Eqs. (23) and (19) with respect to time under the condition that the risk aversion level $\gamma = 2$. It shows that the optimal share of stocks drops dramatically when withdrawals begin and climbs back to the original level as the client gets older. This phenomenon can be explained by the insurance company using 'safer' strategy when it has larger liabilities.

Figures 4 and 5 show the results of problem (17) before retirement. The solid lines are results of Eq.(23). The dotted lines are the results of simply duration matching and the dashed lines are adjusted results after synthesizing wealth maximization and duration matching criteria.

Figures 6 and 7 show the results of problem (17) after retirement. The solid lines are results of Eq.(23). The dotted and dashed lines represent the same meaning as figures 4 and 5.

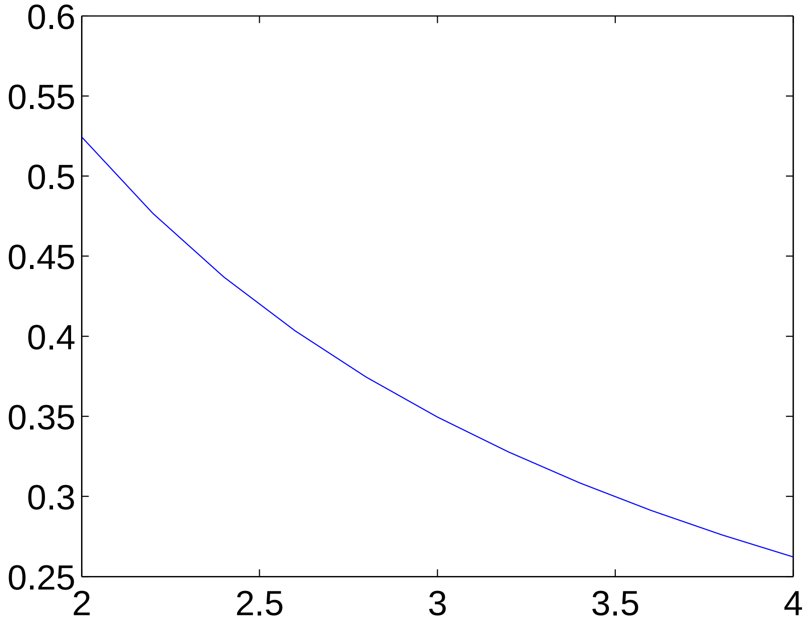

Figure 1: optimal share of risky assets w.r.t. relative risk aversion level

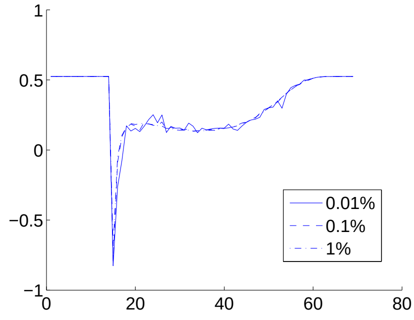

Figure 2: optimal share of risky assets w.r.t. time

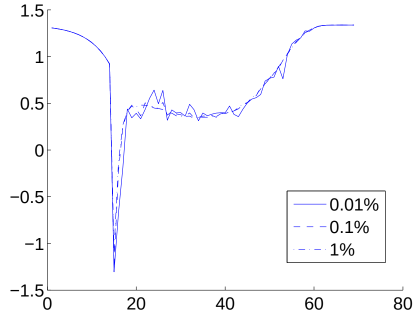

Figure 3: optimal share of bonds w.r.t. time

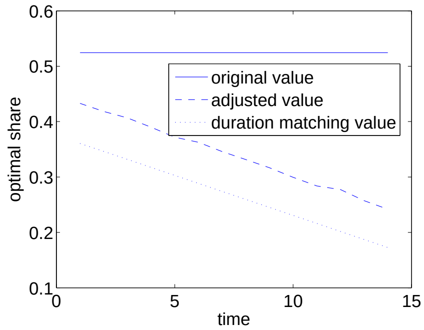

Figure 4: optimal share of risky assets w.r.t. time ($\gamma = 2$,$\alpha = 0.5$)

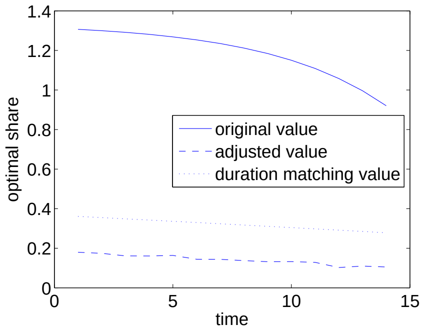

Figure 5: optimal share of bonds w.r.t. time ($\gamma = 2$,$\alpha = 0.5$)

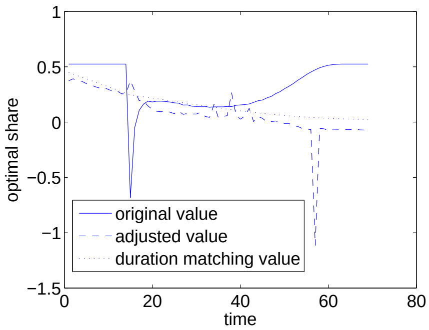

Figure 6: optimal share of risky assets w.r.t. time ($\gamma = 2$,$\alpha = 0.5$)

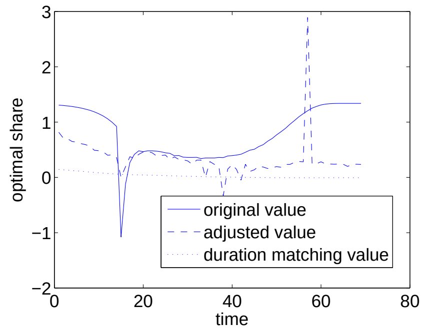

Figure 7: optimal share of bonds w.r.t. time ($\gamma = 2$,$\alpha = 0.5$)

Conclusion

Appendix

Proposition 4.1. *The solution of problem* (14) *is*

$$\begin{cases} \frac{\omega_1^*(t)}{F} = \frac{\sigma_1\lambda_1-\delta_1}{\gamma}, \\ \frac{\omega_2^*(t)}{F} = \frac{\sigma_r(1-\gamma)}{\sigma_K a\gamma}(e^{a(t-T)}-1) - \frac{\sigma_K\lambda_r}{\gamma}. \end{cases}$$

Proof. We can solve problem (14) using section 5.4 in Korn and Korn (2001). Defining

$$v(t,F,r) = \max_\omega \mathbb{E}_t^P\left[\int_0^T U_1(F_s)\psi(s)e^{-\int_0^s \kappa+\psi(u)du}ds + U_1(F_T)e^{-\int_0^T \kappa+\psi(u)du}\right],$$

the HJB equation that corresponds to this problem can be cast in the form of

$$v_t + \frac{F(t)^{1-\gamma}}{1-\gamma}\psi(t)e^{-\int_0^t \kappa+\psi(u)du} + a(b-r(t))v_r + \frac{1}{2}\sigma_r^2 v_{rr}$$
$$+\max_\omega\left\{[F_t r(t) + \omega(t)'(\sigma\lambda-\delta)]\,v_F + \frac{1}{2}\omega'\sigma\sigma'\omega v_{FF} + \omega'\sigma(0\ \sigma_r)'v_{rF}\right\} = 0,$$

with boundary condition

$$v(T,F,r) = \frac{F^{1-\gamma}}{1-\gamma}e^{-\int_0^T \kappa+\psi(u)du}.$$

Formal maximization yields the following candidate for the optimal control:

$$\omega^*(t) = -(\sigma\sigma')^{-1}\frac{\sigma(0\ \sigma_r)'v_{rF} + (\sigma\lambda-\delta)v_F}{v_{FF}}.$$

Inserting ω^* into the HJB equation results in the non linear partial differential equation

$$v_t + \frac{F(t)^{1-\gamma}}{1-\gamma}\psi(t)e^{-\int_0^t \kappa+\psi(u)du} + a(b-r(t))v_r + \frac{1}{2}\sigma_r^2 v_{rr} + F_t r(t)v_F$$
$$-\tfrac{1}{2}(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)\tfrac{v_F^2}{v_{FF}} - \tfrac{1}{2}\tfrac{\sigma_r^2 v_{rF}^2}{v_{FF}} - \tfrac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)'v_F v_{rF}}{v_{FF}} = 0$$

To solve it, we choose the candidate form

$$v(t,F,r) = h(t,r)\,\frac{F^{1-\gamma}}{1-\gamma}e^{-\int_0^t \kappa+\psi(u)du}$$

and obtain a partial differential equation for $h(t,r)$

$$\frac{h_t}{1-\gamma} - \frac{h(\kappa+\psi(t))}{1-\gamma} + \frac{a(b-r)h_r}{1-\gamma} + \frac{1}{2}\frac{\sigma_r^2 h_{rr}}{1-\gamma} + rh + \frac{1}{2}\frac{\sigma_r^2 h_r^2}{h\gamma}$$
$$+\frac{1}{2}(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)\frac{h}{\gamma} + \frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)'h_r}{\gamma} = -\frac{\psi(t)}{1-\gamma}$$

with condition $h(T,r)=1$. We first solve the homogeneous equation associated with this PDE. We guess here that $h(t,r)=e^{A_1(t)+A_2(t)r}$. Thus, we obtain

$$\frac{A_1'(t)+A_2'(t)r}{1-\gamma}-\frac{\kappa+\psi(t)}{1-\gamma}+\frac{a(b-r)A_2(t)}{1-\gamma}+\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{1-\gamma}+r+\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{\gamma}$$
$$+\frac{1}{2}\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)}{\gamma}+\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)'A_2(t)}{\gamma}=0.$$

Rearranging by order of r, we have

$$r\left\{\frac{A_2'(t)}{1-\gamma}-\frac{aA_2(t)}{1-\gamma}+1\right\}+\frac{A_1'(t)}{1-\gamma}-\frac{\kappa+\psi(t)}{1-\gamma}+\frac{abA_2(t)}{1-\gamma}+\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{1-\gamma}$$
$$+\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{\gamma}+\frac{1}{2}\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)}{\gamma}+\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)'A_2(t)}{\gamma}=0.$$

with boundary condition $A_1(T)=A_2(T)=0$. The above equation is equivalent to the following two equations

$$\begin{cases}\frac{A_2'(t)}{1-\gamma}-\frac{aA_2(t)}{1-\gamma}+1=0,\\ \frac{A_1'(t)}{1-\gamma}-\frac{\kappa+\psi(t)}{1-\gamma}+\frac{abA_2(t)}{1-\gamma}+\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{1-\gamma}\\ +\frac{1}{2}\frac{\sigma_r^2A_2^2(t)}{\gamma}+\frac{1}{2}\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)}{\gamma}+\frac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)'A_2(t)}{\gamma}=0.\end{cases}$$

The solution for A_2 is

$$A_2(t)=\frac{1-\gamma}{a}\left[1-e^{a(t-T)}\right].$$

As A_1 has no influence on the optimal share, we omit its solution here. Therefore, we have

$$\frac{\omega^*(t)}{F(t)}=(\sigma\sigma')^{-1}\frac{\sigma(0\ \sigma_r)'A_2(t)+(\sigma\lambda-\delta)}{\gamma}. \tag{23}$$

where the optimal asset allocation $\omega_1^*(t)$ per unit of wealth $F(t)$ is time invariant as its coefficient of $A_2(t)$ is 0 and $\omega_2^*(t)$ per unit of wealth $F(t)$ is time decreasing. We call this ratio the optimal share. To demonstrate it more clearly, we expand the above equation with assumption $\rho_{r1}=0$, and obtain the result in the proposition. □

Proposition 4.2. *The solution to problem (16) is*

$$\omega^*(t)=(\sigma\sigma')^{-1}(F-\sum_{i:t_i>t}w_{t_i}e^{-\int_t^{t_i}r(s)ds})\frac{\sigma(0\ \sigma_r)'A_2+(\sigma\lambda-\delta)}{\gamma}. \tag{24}$$

Proof. To deal with the constraints (15), we put aside at $t=T$ the cash required to satisfy the needs for withdrawals, as in Korn and Krekel (2002). Then, the remaining capital is invested as if there were no withdrawals at all. As F is discontinuous at time t_i, we cannot expect the value function to be continuous at t_i. Therefore, the HJB equation that corresponds to this problem is

$$v_t+\frac{F(t)^{1-\gamma}}{1-\gamma}\psi(t)e^{-\int_0^t\kappa+\psi(u)du}+a(b-r(t))v_r+\frac{1}{2}\sigma_r^2v_{rr}$$
$$+\max_{\omega}\left\{[F_tr(t)+\omega(t)'(\sigma\lambda-\delta)]\,v_F+\frac{1}{2}\omega'\sigma\sigma'\omega v_{FF}+\omega'\sigma(0\ \sigma_r)'v_{rF}\right\}=0,$$

for all $t \in [0, \tau] \backslash \{T+1, \cdots, \llcorner\tau\lrcorner\}$
and

$$v(t_i, F - w_{t_i}) = v(t_i^-, F),$$

for some fixed $t_i = T+1, \cdots, \llcorner\tau\lrcorner$, with the condition that the assets can back the withdrawals during the intermediate process, i.e.,

$$v\Big(t, \sum_{i:t_i>t} w_{t_i} e^{-\int_t^{t_i} r(s)ds}\Big) = 0.$$

Using a process similar to that presented above, we obtain the partial differential equation:

$$\begin{aligned} &v_t + \frac{F(t)^{1-\gamma}}{1-\gamma}\psi(t)e^{-\int_0^t \kappa+\psi(u)du} + a(b-r(t))v_r + \frac{1}{2}\sigma_r^2 v_{rr} + F_t r(t) v_F \\ &-\tfrac{1}{2}(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}(\sigma\lambda-\delta)\tfrac{v_F^2}{v_{FF}} - \tfrac{1}{2}\tfrac{\sigma_r^2 v_{rF}^2}{v_{FF}} - \tfrac{(\sigma\lambda-\delta)'(\sigma\sigma')^{-1}\sigma(0\ \sigma_r)' v_F v_{rF}}{v_{FF}} = 0 \end{aligned}$$

To solve it, we choose the candidate form of

$$v(t,F) = h(t,r)\ \frac{(F - \sum_{i:t_i>t} w_{t_i} e^{-\int_t^{t_i} r(s)ds})^{1-\gamma}}{1-\gamma} e^{-\int_0^t \kappa+\psi(u)du}$$

The procedure to solve $h(t,r)$ is the same except the boundary condition is modified to a transversality condition, resulting $A_2(t) = \frac{1-\gamma}{a}, \quad t > T$.

Thus, we obtain

$$\omega^*(t) = (\sigma\sigma')^{-1}(F - \sum_{i:t_i>t} w_{t_i} e^{-\int_t^{t_i} r(s)ds}) \frac{\sigma(0\ \sigma_r)' A_2 + (\sigma\lambda-\delta)}{\gamma}.$$

where at times t_i we have taken the right-continuous limit of the derivatives. $\square$

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