

On dynamic Minimum Revenue Guarantees in Public Private Partnerships with flexible durations

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Abstract

Public Private Partnerships have been gaining momentum during the last decades as an effective tool for public service delivery. The inefficiency in accurately forecasting the demand and the high incidence of renegotiation are still a matter of concern which undermines the model success. The Least present value auction mechanism can be a solution to neutralize the incidence of the demand risk and renegotiation. The duration of the contract is left flexible and the contract ends once the private operator gets an initially agreed upon value of discounted revenue. This scheme does not, however, provide an efficient mitigation against the potential financial downsides that the project can encounter. This work investigates the incorporation of a minimum revenue guarantee under a Least Present Value Auction mechanism which should improve its hedging power. The guarantee is presented as a multiple exercise Bermudan real option with a variable finite bounded maturity.

keywords: *Public Private Partnerships, Concession, Minimum Revenue Guarantee, Real options, Risk sharing, Multiple exercise option, Least Present Value of revenue auction, compound stopping.*

Introduction

There is an increasing need for public infrastructure delivery either in developed or developing country. According to (McKinsey Global Institute , 2013), this need is roughly estimated at 50\$ trillion for the period between 2013 and 2030. Public Private Partnerships (PPPs) can be an interesting tool to help governmental agencies to cope with their sharp needs. They can be seen as a hybrid form of privatization and traditional public procurement. They help mobilizing the private expertise and skills in the delivery of public infrastructure which is reputed to increase efficiency and reduce costs. They are generally conducted under a project finance framework where a special purpose vehicle (SPV) is created and is in charge of all the activities related to the project. The SPV is financed by equity and non-recourse debt with a very high leverage that ranges between 70% to 95% (Yescombe, 2011). The SPV has to design, finance, construct and operate the infrastructure for a certain period. In return for the services provided, the private operator has the right to charge fees for the use of the infrastructure (This contractual form can be referred to in some countries such as France as concession contracts). The whole transaction hinges in the future cash flow that the SPV is entitled to collect. At the end of the contract,

the infrastructure reverts back to the public entity. The most appealing strength of PPPs is the principle of risk sharing that governs the design of the transaction and which is reputed to verify the optimal value for money. In principle, each party should bear the risk that she is best able to manage its consequence and mitigate the chance of its occurrence. The demand risk is a striking exception for the previous principle since the usage level incorporates many systematic and natural influences which are out of the control of the stakeholders. The question on who should support it is still an open query. Traditionally, the demand risk is supported by the private operator but the high occurrence of renegotiation¹ and the difficulty of accurately estimating the demand ²has led to the emergence of a recent trend which promotes for the allocation of the demand risk totally to the public entity. The contracting authority commits to provide a payment contingent on the availability of the infrastructure and the private entity is in charge of collecting fees on its behalf. This scheme reduces, however, the private incentive to increase the commercial appeal of the project (Ellman, 2006; Athias and Soubeyran, 2013; Iossa, 2015) . The demand can, consequently, decrease which may increase the public exposure to risk. The recent devel-

¹ see, for instance, (Guasch, 2004; Guasch et al., 2007, 2006; Guasch and Straub, 2006)

²see, for instance, (Flyvbjerg et al., 2005; Bain, 2009; Nicolaisen et al., 2012)

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opments on the study of optimal allocation of demand risk in public private partnerships suggest that PPP contracts should be flexible in order to allow potential future adjustments of the economic balance of the contract once more accurate information about the demand is revealed over time. Flexible contracts permit to share the demand risk between the two parties which should boost collaboration (Engel et al., 2009; Athias and Saussier, 2007; Dong and Chiara, 2010; Iossa, 2015). Flexibility allows, in fact, to incorporate a structure of compensations in the initial contractual terms which are offered to all bidders at the tendering process. This allows to increase the transparency of the contractual framework and to take into account the future chance of potential renegotiation. In other words, the implicit guarantee of potential renegotiation that the public entity commits to, due to the high political sensitivity of infrastructure projects and the high pressure that public deciders can face, if the service is interrupted, is made implicit.

One of the most known forms of flexible design in Public Private Partnerships is the Least Present value mechanism proposed by (Engel et al., 1997, 1998, 2002, 2003). Instead of being based on the tariff charged to end users or on the contract duration, bids consist in proposing a value for the project which is expressed in present value of revenue based on a pre-established discounting rate and tariff. The contract duration is left flexible until the required cumulative discounted sum of revenue is collected. Afterwards, the project reverts to the public entity. The contract's term changes, therefore, under different scenarios of demand which permits to keep a constant present value of revenue among all the different states of nature. The contract specifies additionally a maximal period T_{max} for which the concession can last, even if the targeted LPVR is not reached. The LPVR mechanism can be seen as a contract in which renegotiation occurs continuously in the favor of one party or another depending on the level of demand. The scheme does not, nevertheless, totally guarantee the financial viability of the project since the revenue level may be insufficient for the SPV to meet all its legal obligations. The discounted sum of revenue may be not, moreover, recouped at T_{max} especially for projects with a high volatility which increases the contractors risk.

One solution to the previous two pitfalls is to combine the contract with a minimum revenue guarantee (MRG) which is a common approach to mitigate the demand risk under traditional concessions. The government guarantees to the SPV the revenue shortfall for a certain number of years on the basis of a pre-defined minimum revenue guaranteed. MRGs can be structured either under a European structure or an American structure³. Under the former contract, the potential dates are chosen before the beginning of the contract. The latter contract leaves to the SPV the freedom to choose her poten-

tial exercise dates which permits to take advantage from information revealed over time. The American contract offers more flexibility and is easier to manage since it is impossible to determine beforehand the best exercise dates at which the guarantee should be redeemed.

This work proposes to combine the standard LPVR scheme with an American minimum revenue guarantee. This should increase the bankability of the project and reduce the chance of not collecting the agreed-upon discounted sum of revenue. This contingent liability can be seen as an inter-temporal transfer of wealth between the public entity and the SPV since the potential payments will be recouped once the project is transferred back to the public entity. Valuing the guarantee is essential for both sponsors and public deciders in order to know the extent to which risk is being transferred and for the optimal design of the contract. It is also crucial for budgetary purposes in order to provide adequate provision and to measure the fiscal risk that the public entity faces.

The guarantee, which is denoted MRG-LPV, can be treated as a Bermudan option with a changing unknown maturity since claiming a compensation at a certain date may lead to a reduction of the contract duration. For a multiple exercise contract, there is a dependence of the reward on previous actions. The decision maker has to remember, at a given date, her previous actions and the reward is, consequently, non-Markovian. The contract valuation can be tackled as a discrete time compound stopping problem and embedded dynamic programming can be used to derive an approximate exercise policy. This approach turns out to be computationally extensive and suffers from the curse of dimensionality for long-maturities and contracts with a considerable number of exercise dates. We propose, therefore, an approach to simplify the valuation procedure which permits to have a faster derivation of an approximate value of the MRG-LPV.

This work presents, additionally, a framework which allows efficient risk management. For this aim, the valuation is undertaken under the real world measure in order to have understandable and real distributions of the project's financial indicators. The determination of the project's valuation is made by Monte Carlo simulation and by following the Marketed Asset Disclaimer of (Copeland and Antikarov, 2003).

The remaining of this development unfolds as follows: Section 1 presents how the valuation of the real option can be made under the real-world measure. Section 2 presents the valuation procedure for the contract with a single right. Section 3 extends the valuation problem to a multiple exercise setting. A faster approximate valuation procedure is presented in Section 4. A numerical illustration via a hypothetical yet realistic project is made in section 5. A conclusion is drawn in section 6.

³European guarantees can be seen as a portfolio of standard European options (Irwin, 2007; Brandao and Saraiva, 2008). American guarantees can be seen as a swing option (Jaillet et al., 2004; Chiara et al., 2007)

1 Real-world valuation of real options

The most challenging problem in real option valuation from both a theoretical and practical levels is the determination of a market consistent value. For this purpose, one should know the appropriate discounting factor at which the cash flows that the option leads to should be deflated. The most straightforward approach is to use the risk neutral probability as for financial options. The key input is the underlying asset volatility. In the context of real assets, volatility may be vague, difficult to observe, measure and apprehend. This is particularly true given the scarcity of historical data⁴. What is, for instance, the volatility of an R&D project or what is the volatility of a toll road? Monte Carlo simulation can be used to overcome this conceptual limitation. This approach has become accepted in the general context of real option and in the particular context of public private partnerships⁵. It was, initially, conceptualized in (Copeland and Antikarov, 2003). The basic idea behind this method is to simulate the rate of return of the project and derive its volatility. This mechanism is mainly founded on two major assumptions. The first one is that properly anticipated returns fluctuate randomly as demonstrated by (Samuelson, 1973). This implies that independently of the nature of the cash flows, the evolution of the project's present value follows a random walk. This allows reducing any combination of complex uncertainties to a sole stochastic process which is the project's value variability over time. This value is denoted PV_t . The second assumption is referred to as the *Marketed Asset Disclaimer* which considers the project without flexibility as the best estimate for the twin security for the contingent claim on the project and treats it as if it was a traded asset. Such value is determined in association with the discounted cash flow analysis.

In the remaining of this work, we follow (Copeland and Antikarov, 2003) and consider a complete market with two assets: a risk free Bank account B_t solution of $dB_t = rB_t dt$ and the project without flexibility as a risky asset which has a value PV_t at year t . We assume here that the project's present value follows a standard geometric Brownian motion with a changing volatility to account for the potential changes in operational leverage (Brandão et al., 2012):

$$dPV_t = r_e PV_t dt + \sigma_t PV_t dW_t, \quad (1)$$

where r_e is the expected cost of equity. σ_t is the volatility of the project at period t and which can be determined by simulating the different sources of uncertainties in a Monte Carlo simulation and computing the variability of the logarithmic

⁴The World Bank PPI database reports, for instance, that in developing countries more than 60% of the PPP projects in were greenfield projects with no historical record at all

⁵see for instance (Herath and Park, 2002; Mun, 2002; Nembhard et al., 2003; Brandão et al., 2005a,b; Smith, 2005) in the general context and (Brandão and Saraiva, 2008; Ashuri et al., 2011) for public private partnerships

return $y_t = \log\left(\frac{PV_t}{\mathbb{E}[V_0]}\right)$ of the project's present value:

$$\sigma_t = \sqrt{\frac{y_t}{t}}. \quad (2)$$

PV_t is determined simultaneously with the stochastic discounted cash flow analysis of the project. It is given by the expected Adjusted Net Present Value at t conditional on the already known information at that time (Brealey, 2012). The derivation of the conditional expectation within the Monte Carlo simulation can be made by the means of the Least Squares Monte Carlo (LSM) technique of (Longstaff and Schwartz, 2001) as in (Godinho, 2006).

At this point, one can switch to the risk neutral measure and derive the value of a given real option written on the project's value. The main limitation of the risk neutral measure is that only the expectation can be understood by a decision maker in the real world. Performing a reliable risk analysis and optimization becomes impossible due to the deformation of all the probabilities. The whole effect of the real option on the project's risk profile cannot be identified since the determined probabilities have no meaning under the real world measure. For instance, one can derive the expected net present value of the project in the presence of a certain guarantee but cannot know exactly the chance of having a negative net present value. This can be problematic to the design of flexibilities which aim to appropriately share the risk between the different stakeholders of the project. Under the risk neutral measure, decision makers are not able to know how much risk they are taking in their purses. Moreover, the models under the risk neutral measure, cannot be appropriately used for budgeting by public entities due to the deformation of probabilities. A public decider cannot, for example, say what is the chance for the guarantee to go unused and the extent of his fiscal exposure. One requires, therefore, an approach to derive a market consistent value while working under the natural probability measure. A "bridge" which allows to make the connection between the two worlds is, then, needed. The existence of the so-called bridge is guaranteed by the law of one price. In complete markets, there is a unique and positive process ξ_t known as the stochastic discounting factor (SDF) which allows to value a sequence of stochastic cash flows $x = (x_1, \dots, x_n)$:

$$p_t = \frac{1}{\xi_t} \sum_{i=t}^n \mathbb{E}_t[x_i \xi_i]. \quad (3)$$

where p_t is the fair price of x at a given time t (Cochrane, 2009). This process represents the link between the two worlds since to the left side of equation (3), we have the "real" value of the contract and on the right side we have the real world measure. To determine the adequate stochastic discounting factor for the project, we follow (Duffie, 2010).

For any probability measure \mathbb{Q} equivalent to \mathbb{P} , one can define the density process π for \mathbb{Q} which is the martingale

defined by:

$$\pi_t = \mathbb{E}_t \left[\frac{dQ}{dP} \right], \quad (4)$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P . For any random variable X such that $\mathbb{E}^Q[X] < \infty$:

$$\mathbb{E}_t^Q[X] = \frac{\mathbb{E}_t^P[X]}{\pi_t}. \quad (5)$$

The martingale representation theorem gives that the π_t can be written as follows:

$$d\pi_t = -\kappa_t \pi_t dW_t^P. \quad (6)$$

To construct the stochastic deflator, one can use the following result from (Duffie, 2010) which states that: *If in the existence of a short-rate process r and the definition of a bank account $B_t = \exp\left(\int_0^t r_s ds\right)$ and after a deflation by B , there is an equivalent martingale measure with a density process π_t , then a state price deflator ξ is defined by $\xi_t = \pi_t B_t$, provided $\text{var}(\pi_t) < \infty$ for all t .*

All what one has to do to derive the stochastic discounting factor is to determine the state density of the risk neutral measure. This is given by the Girsanov theorem which states that in order for Q to be the risk neutral probability for the project value, κ_t should be equal to the market price of risk λ_t :

$$\lambda_t = \frac{r_e - r}{\sigma_t}. \quad (7)$$

The stochastic deflator ξ_t is then given by:

$$\xi_t = \exp\left(-\left(r + \frac{\lambda_t^2}{2}\right)t - \lambda_t W_t\right), \quad (8)$$

with the convention that $\xi_0 = 1$.

2 The one-right contract

Let PVR_0 denote the targeted cumulative sum of discounted revenue and PVR_t denote the cumulative discounted value of revenue for each year $t = 1, \dots, T_{max}$:

$$PVR_t = \min\left(PVR_0, \sum_{i=1}^t \frac{R_i}{(1+r_c)^i}\right), \quad (9)$$

where R_t is the revenue process of the project and r_c is the pre-established discounting factor by the public entity used for the discounting for the present values of revenue. The contract ends in all the states of nature at most at T_{max} , even if the private entity was not able to collect PVR_0 . Figure 1 presents the mechanism of the contract for different levels of demand. The contract duration T^* can be determined as follows:

$$T^* = \min\left\{T_{max}, \inf\{k \mid PVR_k \geq PVR_0\}\right\}. \quad (10)$$

The terminal present value of revenue is denoted PVR^* and is given by:

$$PVR^* = \min(PVR_0, PVR_{T^*}). \quad (11)$$

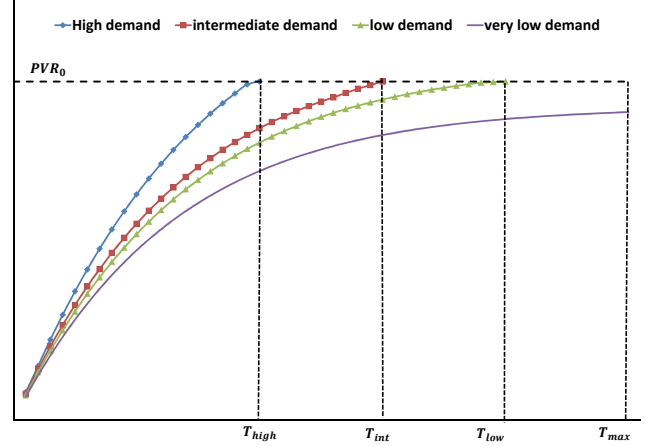


Figure 1: The contract structure under different scenarios of demand

Under a one right LPV-MRG contract, the public entity grants the SPV the revenue shortfall Z_t , if the revenue falls below a threshold $\{K_t\}_{t=1}^{T_{max}}$, and if she has not yet collected PVR_0 . Let G_t denote the traditional compensation for a Minimum Revenue Guarantee for each $t = 1, \dots, T_{max}$.

$$G_t = \max(0, K_t - R_t). \quad (12)$$

This compensation matches the compensation of a standard put on the underlying real asset R_t with an exercise strike of K_t . We introduce, additionally, the discounted reward G_t^c by the means of the established discounting rate r_c defined by:

$$G_t^c = \frac{G_t}{(1+r_c)^t}. \quad (13)$$

The compensation of the Minimum Revenue Guarantee under a Least Present Value of Revenue Z_t is defined as follows for each year t :

$$Z_t = \begin{cases} G_t, & \text{if } G_t^c + PVR_t \leq PVR_0, \\ (PVR_0 - PVR_t)(1+r_c)^t, & \text{otherwise.} \end{cases} \quad (14)$$

We introduce the quantity H_t defined as follows:

$$H_t = \min(G_t^c, PVR_0 - PVR_t). \quad (15)$$

Equation (14) can be simplified to:

$$Z_t = H_t (1+r_c)^t. \quad (16)$$

We introduce an exercise cemetery $\partial := T_{max} + 1$ at which the right is necessarily exercised and at which the reward is null: $Z_\partial = G_\partial := 0$. We take the additional convention that $PVR_\partial := PVR_0$. The payoff is extended beyond the contract terminal duration T^* , even though the contract cannot be exercised after T^* , for the convenience of the presentation and the reasoning later on. For any $t > T^*$, the compensation is null by definition. Claiming Z_t does, additionally, impact the contract duration which is shortened. To capture this effect, we introduce a corrected real contract duration $T^*(H_t)$ after a discounted payment H_t is made:

$$T^*(H_t) = \inf \left\{ k \mid H_t + PVR_t \geq PVR_0 \right\}. \quad (17)$$

The contract can be structured in a European style where the SPV selects the exercise date prior to the beginning of the contract. In that case the value of the contract V_0^{static} is given by:

$$V_0^{static} = \mathbb{E}_0 [Z_{t_e}^d], \quad (18)$$

where t_e is the chosen exercise date and $Z_t^d = \xi_t Z_t$ is the discounted compensation by the means of the stochastic discounting factor ξ_t . The deflator is taken null after the termination of the contract (i.e $\xi_t = 0, \forall t > T^*$). When computing the volatility of the project, any path for which the contract has been already terminated is omitted from the Monte Carlo simulation, we keep only the paths at which the contract is still existing at a given time step t . In order to avoid any confusion and simplify the presentation, we discount all the cash flows to the original time.

This development focuses on the American style contract because it guarantees the maximal flexibility in the contract management⁶. Under this contract, the decision maker chooses, during the contract life, when she will claim her right. Her exercise strategy is adapted to the revenue level that she sees and she can take, therefore, advantage from the information revealed over time. In this context, the contract becomes similar to a Bermudan option with an unknown finite bounded maturity T^* . The contract fair value is equal to the maximal compensation that the private entity can obtain by the means of a non anticipative exercise policy. The decision maker chooses an exercise date (stopping time with respect to the natural filtration \mathcal{F}_t) in order to maximize her return. The contract fair value V_t at a time t is, therefore, solution of the following optimization problem:

$$V_t = \max_{\tau \in \{t, \dots, T^*\} \cup \{\partial\}} \mathbb{E}_t [Z_\tau^d]. \quad (19)$$

where \mathbb{E}_t denotes the conditional expectation at time t with respect to the natural filtration \mathcal{F}_t . Since the payoff is extended beyond T^* and is null by definition for any $T^* < t \leq T_{max}$, V_t is, also, solution of:

$$V_t = \max_{\tau \in \{t, \dots, T_{max}\} \cup \{\partial\}} \mathbb{E}_t [Z_\tau^d]. \quad (20)$$

⁶refer to (Chiara et al., 2007) for an argument.

The real option is equivalent to a Bermudan option with a maturity T_{max} and a compensation Z_t . The extension of the payoff beyond T^* allows to overcome any additional computation that may arise because of the variability of the contract duration for different scenarios. Let t denote, in the remaining of this development, a date at which the valuation is being made. If the right was claimed prior to t , the decision maker has no choice but continuing until the terminal date of the contract T^* . Otherwise, she chooses among two decisions:

- either **exercise** and claim the compensation Z_t , the contract final duration changes immediately to $T^*(H_t)$,
- or **continue** and hold the same contract where she can exercise starting from the next time step $t + 1$. The contract duration does not change in that case.

We introduce the continuation value $Q_t = \mathbb{E}_t [V_{t+1}]$ which measures the expected reward, if no exercise is made. The contract value can be determined recursively by the means of the following Bellman equation:

$$\begin{cases} V_{T_{max}} = Z_{T_{max}}^d, \\ V_t = \max \{ Z_t^d, Q_t \}, \end{cases} \quad (21)$$

An exercise indicator I_t can be defined as follows:

$$I_t = \begin{cases} 1, & \text{if } Z_t^d \geq Q_t, \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

where 0 stands for continuation and 1 for exercising and we set $I_\partial := 1$. The optimal exercise date τ_t is the first date at which the strategy indicates to exercise and is obtained as follows:

$$\tau_t = \inf \left\{ k \geq t \mid I_k = 1 \right\}. \quad (23)$$

The contract fair value is then given by:

$$V_t = \mathbb{E}_t [Z_{\tau_t}^d]. \quad (24)$$

Computing the continuation value under a Monte Carlo simulation can be very time consuming. One can rely on an approximation procedure to determine a lower bound on the contract value. If \tilde{Q}_t is an approximation of the real continuation value (e.g by the means of the Least-Square Monte Carlo (LSM) proposed by (Longstaff and Schwartz, 2001)), a near optimal stopping time $\tilde{\tau}_t$ can be derived and a lower bound on the contract can be obtained by:

$$\underline{V}_t = \mathbb{E}_t [Z_{\tilde{\tau}_t}^d]. \quad (25)$$

3 The multiple exercise contract as a compound stopping problem

This section extends the MRG-LPVR contract to a multiple exercise setting. Under a least present value of revenue scheme,

claiming a compensation does immediately change the structure of the contract (the duration may be shortened). The problem becomes non-Markovian, since the immediate reward depends on previous actions and the decision maker has therefore, to remember the dates at which she has claimed her prior rights. She has also to anticipate the dates at which the future reward will be claimed in order to make her decision at the present time. This dependence on the reward structure among all the stopping times adds a layer of complexity to the valuation procedure of minimum revenue guarantees under a least present value of revenue scheme in comparison with traditional multiple-exercise Bermudan options. One solution to value the contract is to consider the valuation problem as a compound stopping problem. Under this setting, the decision maker can remember the dates at which the several rights are claimed and adapts her decision accordingly.

3.1 The two-right contract

We assume, first, that the contract offers two exercise rights. Let t denote a time at which a decision has to be made. Assuming that there are no claimed compensations yet, the decision maker starts by choosing the date at which she will claim her first right $\tau_t^{(1)} \geq t$ which is a stopping time according to the natural filtration \mathcal{F}_t . The compensation $Z_{\tau_t^{(1)}}^{(1)}$ that she gets is given by:

$$Z_{\tau_t^{(1)}}^{(1)} = H_{\tau_t^{(1)}} (1 + r_c)^{\tau_t^{(1)}}. \quad (26)$$

This compensation matches the compensation of the one right contract presented in section 2. The choice of $\tau_t^{(1)}$ does impact the structure of the contract since the cumulative discounted revenue is increased by the amount of $H_{\tau_t^{(1)}}$ for all the remaining time horizon and the contract duration may be shortened consequently. To capture this effect, one needs to introduce a new filtration $\mathcal{F}_{\tau_t^{(1)},s}$ which contains all the information available at a time $s \geq \tau_t^{(1)}$, knowing that the first right was claimed at time $\tau_t^{(1)}$. One has $\mathcal{F}_{\tau_t^{(1)}} \subseteq \mathcal{F}_{\tau_t^{(1)},s}$. The decision maker chooses, afterwards, a second date $\tau_{t|\tau_t^{(1)}}^{(2)}$ at which she will claim her second right. To simplify the presentation $\tau_{t|\tau_t^{(1)}}^{(2)}$ will be denoted $\tau_t^{(2)}$ and the dependence towards $\tau_t^{(1)}$ is implicitly assumed. $\tau_t^{(2)}$ is, here, a stopping time with respect to the new filtration $\mathcal{F}_{\tau_t^{(1)},s}$. The compensation $Z_{\tau_t^{(2)}}^{(2)}$ that she gets for her second exercise right knowing that the first right was claimed at $\tau_t^{(1)}$ is given by:

$$Z_{\tau_t^{(2)}}^{(2)} = \max \left(0, \min \left(G_{\tau_t^{(2)}}^c, PVR_0 - PVR_{\tau_t^{(2)}} - G_{\tau_t^{(1)}}^c + G_{\tau_t^{(2)}}^c \right) \right) (1 + r_c)^{\tau_t^{(2)}}. \quad (27)$$

We introduce two exercise cemeteries $\partial^{(1)} =: T_{max} + 1$ and $\partial^{(2)} =: T_{max} + 2$ at which respectively, the first and

second rights are necessarily exercised. The exercise cemeteries verify : $Z_{\partial^{(1)}}^{(1)} := 0$ and $Z_{\partial^{(2)}}^{(2)} := 0$. We set moreover $PVR_{\partial^{(1)}} = PVR_{\partial^{(2)}} := PVR_0$.

The contract total reward is denoted $Z_{\{\tau_t^{(1)}, \tau_t^{(2)}\}}$ and is defined by:

$$Z_{\{\tau_t^{(1)}, \tau_t^{(2)}\}} = Z_{\tau_t^{(1)}}^{(1)} + Z_{\tau_t^{(2)}}^{(2)}. \quad (28)$$

The set of admissible exercise policies $\Pi_t^{(2)}$ at time t is

$$\Pi_t^{(2)} = \left\{ (t_1, t_2) \in \{1, \dots, \partial^{(1)}\} \times \{1, \dots, \partial^{(2)}\} \right. \\ \left. t \leq t_1 \leq \partial^{(1)}, t_1 < t_2 \leq \partial^{(2)} \right\}. \quad (29)$$

Let $\mathbb{E}_m, \mathbb{E}_{m,n}$ denote respectively the conditional expectation with respect to \mathcal{F}_m and $\mathcal{F}_{m,n}$. The contract fair value $V_t^{(2)}$ is the maximal compensation that the decision maker can get via a non-anticipative admissible exercise policy which can be defined as follows:

$$V_t^{(2)} = \max_{(\tau_t^{(1)}, \tau_t^{(2)}) \in \Pi_t^{(2)}} \mathbb{E}_t \left[Z_{\{\tau_t^{(1)}, \tau_t^{(2)}\}}^d \right], \quad (30)$$

where $Z_{\{\tau_t^{(1)}, \tau_t^{(2)}\}}^d$ is the discounted reward defined as follows:

$$Z_{\{\tau_t^{(1)}, \tau_t^{(2)}\}}^d = \xi_{\tau_t^{(1)}} Z_{\tau_t^{(1)}}^{(1)} + \xi_{\tau_t^{(2)}} Z_{\tau_t^{(2)}}^{(2)} \quad (31)$$

The valuation problem is a compound stopping problem. In general a compound stopping variable is a pair of random variables (τ_1, τ_2) with values in $\Pi_t^{(2)}$ and which satisfy the following properties (Haggstrom, 1966):

1. $\tau_1 < \tau_2$ a.s,
2. $\{\tau_1 = m\} \in \mathcal{F}_m$ for all $m \geq 1$,
3. $\{\tau_1 = m, \tau_2 = s\} \in \mathcal{F}_{m,s}$ for all $n > m \geq 1$.

To determine the optimal exercise policy, the decision maker can rely on dynamic programming. For each time t , when she has not claimed any right yet, she chooses among two decisions :

- Continue and hold the same contract where she can exercise her right starting from the next time $t + 1$. Her reward in that case is measured by the expected value of her contract at the next time step: $Q_t^{(2)} = \mathbb{E}_{t+1} [V_{t+1}^{(2)}]$,
- Exercise and set the first stopping time to t (i.e $\tau_t^{(1)} = t$). Her reward is measured by the following conditional expectation $Q_t^{(1)} = \mathbb{E}_t [L_{t,t+1}]$, where: $L_{t,l} = \max_{l \leq m \leq T_{max}} \mathbb{E}_{t,l} [Z_{t,m}]$. $Q_t^{(1)}$ measures the expected

reward, if the first exercise right is claimed at t , and if the second right is exercised optimally. Determining $H_{t,t+1}$ requires an embedded dynamic programming procedure. At the terminal date T_{max} , one has $L_{t,T_{max}} = Z_{\{t,T_{max}\}}^d$. Starting from this point the decision maker chooses, for each decision time m , $t < m < T_{max}$:

- either to continue and hold the contract with one remaining right knowing that she has exercised at t and where she can exercise starting from the next date $m + 1$. Her reward is, then, measured by $Q_m^{(1,0)}(t) = \mathbb{E}_{t,m+1} [L_{t,m+1}]$,
- or to exercise and get the reward $Z_{\{t,m\}}^d$.

The previous reasoning can be formalized as follows:

$$\left\{ \begin{array}{l} L_{\{t,T_{max}\}} = Z_{\{t,T_{max}\}}^d, \quad t = 1, \dots, T_{max}, \\ L_{\{t,m\}} = \max \left(Z_{\{t,m\}}^d, \mathbb{E}_{t,m} [L_{\{t,m+1\}}] \right), \\ m = t + 1, \dots, T_{max} - 1, \\ V_{T_{max}}^{(2)} = Z_{\{T_{max}, \partial^{(2)}\}}^d \\ V_t^{(2)} = \max \left(\mathbb{E}_t [L_{\{t,t+1\}}], \mathbb{E}_t [V_{t+1}^{(2)}] \right), \\ t = 1, \dots, T_{max} - 1. \end{array} \right. \quad (32)$$

An exercise indicator $I_t^{(1)}$ for the first stopping time can be determined as follows:

$$I_t^{(1)} = \begin{cases} 0, & \text{if } \mathbb{E}_t [V_{t+1}^{(2)}] > \mathbb{E}_t [L_{\{t,t+1\}}] \\ 1, & \text{otherwise,} \end{cases} \quad (33)$$

and we set $I_{\partial^{(1)}}^{(1)} := 1$.

An exercise indicator $I_{t|m}^{(2)}$ for the second stopping time knowing the first one to be m is given as follows for each $t > m$ by:

$$I_{t|m}^{(2)} = \begin{cases} 0, & \text{if } Z_{\{m,t\}} < \mathbb{E}_{m,t} [L_{\{m,t+1\}}] \\ 1, & \text{otherwise} \end{cases} \quad (34)$$

and we set $I_{\partial^{(2)}|\tau}^{(2)} := 1, \quad \forall \tau \in \{1, \dots, \partial^{(1)}\}$.

The optimal exercise policy at given time t can be obtained as follows:

$$\left\{ \begin{array}{l} \tau_t^{(1)} = \inf \left\{ k \geq t, \quad | \quad I_k^{(1)} = 1 \right\}, \\ \tau_t^{(2)} = \inf \left\{ k > \tau_t^{(1)}, \quad | \quad I_{k|\tau_t^{(1)}}^{(2)} = 1 \right\}. \end{array} \right. \quad (35)$$

A lower bound on the contract value can be obtained by approximating the different continuation values by the LSM

technique which yields an approximate compound stopping strategy $(\tilde{\tau}_t^{(1)}, \tilde{\tau}_t^{(2)})$. A lower bound on the contract value $\underline{V}_t^{(2)}$ is given by:

$$\underline{V}_t^{(2)} = \mathbb{E}_t \left[Z_{\{\tilde{\tau}_t^{(1)}, \tilde{\tau}_t^{(2)}\}} \right]. \quad (36)$$

3.2 The multiple-exercise contract

We generalize the two-right contract to a multiple exercise setting. The public entity offers to the SPV the possibility to claim the compensation at n occasions. Let $(\tau)_i = (\tau_1, \dots, \tau_i)$ denote an exercise policy for the first i rights. The compensation that the SPV can get for her i^{th} right at a given time t knowing that the previous rights were exercised at $(\tau)_{i-1}$ is given by:

$$\left\{ \begin{array}{l} Z_{t|\emptyset}^{(1)} = \max \left(0, \min \left(G_t^c, LPV_0 - LPV_t - \right. \right. \\ \left. \left. - G_t^c \right) \right) (1 + r_c)^t \\ Z_{t|(\tau)_{i-1}}^{(i)} = \max \left(0, \min \left(G_t^c, LPV_0 - LPV_t - \right. \right. \\ \left. \left. \sum_{j=1}^{i-1} G_{\tau_j}^c - G_t^c \right) \right) (1 + r_c)^t, \quad i = 2, \dots, n. \end{array} \right. \quad (37)$$

The overall compensation $Z_{\{(\tau)_n\}}$ for a family of exercise dates (τ_1, \dots, τ_n) is defined as follows:

$$Z_{\{(\tau)_n\}} = \sum_{i=1}^n Z_{t|(\tau)_{i-1}}^{(i)}. \quad (38)$$

The discounted reward $Z_{\{(\tau)_n\}}^d$ is given by:

$$Z_{\{(\tau)_n\}}^d = \sum_{i=1}^n Z_{\tau_i|(\tau)_{i-1}}^{(i)} \xi_{\tau_i} \quad (39)$$

The value of the static contract is straightforward and is given by:

$$V_0^{static}(n) = \mathbb{E}_0 \left[Z_{\{t_e^{(1)}, \dots, t_e^{(n)}\}}^d \right], \quad (40)$$

where $(t_e^{(1)}, \dots, t_e^{(n)})$ is the chosen family of exercise dates prior to the beginning of the contract. Not all the rights can be exercised since the contract may end before some of the selected dates are reached.

For the American style contract, the problem becomes an n-compound stopping problem. A family (τ_1, \dots, τ_n) is called an n -multiple stopping rule if the following conditions hold (Mandelbaum and Vanderbei, 1981; Sofronov et al., 2006):

- $\tau_1 \leq \dots \leq \tau_i$ a.s,
- $\{\tau_1 = m_1, \dots, \tau_i = m_i\} \in \mathcal{F}_{m_1, \dots, m_i}$
for all $m_i > m_{i-1} > \dots > m_1 \geq 1, \quad i = 1, 2, \dots, n$

Here $\mathcal{F}_{m_1, \dots, m_i}$ contains all the information up to the time m_i dynamic program: knowing that the first stops were made at $(m)_{i-1} = (m_1, \dots, m_{i-1})$.

The guarantee fair value is given by :

$$V_t^{(n)} = \sup_{(\tau)_{n,t} \in \Pi_t^{(n)}} \mathbb{E}_t \left[Z_{\{\tau\}_{n,t}}^d \right], \quad (41)$$

where $(\tau)_{n,t}$ is a n - compound stopping variable and $\Pi_t^{(n)}$ is the set of admissible n - compound stopping variables at time t defined as follows:

$$\begin{aligned} \Pi_t^{(n)} = & \left\{ (t_1, \dots, t_n) \in \left\{ t, \dots, T_{\max}, \partial^{(1)} \right\} \times \dots \times \right. \\ & \left. \left\{ t, \dots, T_{\max}, \partial^{(n)} \right\} \mid t < t_1 \leq \partial^{(1)}, \right. \\ & \left. t_{j-1} < t_j \leq \partial^{(j-1)}, j = 2, \dots, n \right\}, \end{aligned} \quad (42)$$

where $\partial^{(i)}$, $i = 1, \dots, n$ denotes the exercise cemetery for the i^{th} right. Let $\Pi_{t|(m)_i}^{(n-i+1)}$ denote the set of feasible exercise policies knowing the feasible sequence $(m)_i$ at time t (i.e $t \leq m_1 < \dots < m_i$ and $m_j \leq \partial^{(j)}$, $j = 1, \dots, i$), it is defined as follows:

$$\left\{ \begin{aligned} \Pi_{t|(m)_i}^{(n-i)} = & \left\{ (t_{i+1}, \dots, t_n) \in \left\{ 1, \dots, \partial^{(i+1)} \right\} \times \dots \times \right. \\ & \left. \left\{ 1, \dots, \partial^{(n)} \right\} \mid m_i < t_1 \leq \partial^{(1)}, \right. \\ & \left. t_{j-1} < t_j < \partial^{(j-1)}, j = i+2, \dots, n \right\}, \\ \Pi_{t|\emptyset}^{(n-i)} = & \Pi_t^{(n-i)}. \end{aligned} \right. \quad (43)$$

We introduce the value $V_{t|(m)_i}^{(n-i)}$ of the contract at time t knowing that the first i stops were made at the sequence $(m)_i$ defined as follows:

$$V_{t|(m)_i}^{(n-i)} = \sup_{(\tau)_{n-i} \in \Pi_{t|(m)_i}^{(n-i)}} \mathbb{E}_{(m)_i} \left[Z_{\{(m)_i, (\tau)_{n-i}\}}^d \right], \quad (44)$$

with the convention that $V_{t|\emptyset}^{(n)} = V_t^{(n)}$ and $V_{t|(m)_n}^{(0)} = Z_{\{(m)_1, \dots, (m)_n\}}^d$

We define an exercise indicator $I_{t|(m)_{i-1}}^{(i)}$ knowing the sequence $(m)_i$ at time t for each $i = 1 \dots, n$ as follows:

$$I_{t|(m)_{i-1}}^{(i)} = \begin{cases} 1, & \text{if } \mathbb{E}_{(m)_{i-1}} \left[V_{t|\{(m)_{i-1}, m_{i-1}+1\}}^{(n-i)} \right] \geq \\ & \mathbb{E}_{(m)_{i-1}} \left[V_{t|\{(m)_{i-1}, m_{i-1}+1\}}^{(n-i+1)} \right], \\ 0 & \text{otherwise,} \end{cases} \quad (45)$$

and we set $I_{\partial^{(i)}|(m)_{i-1}}^{(i)} = 1$, $\forall (m)_{i-1}$.

The contract value can be obtained by the following recursive

$$\begin{aligned} V_{t|(m)_i}^{(n-i)} = & V_{t|\{(m)_{i-1}, m_i+1\}}^{(n-i)} (1 - I_{t|(m)_i}^{(i)}) + \\ & V_{t|\{(m)_i, m_i+1\}}^{(n-i-1)} I_{t|(m)_i}^{(i)}. \end{aligned} \quad (46)$$

The sequence of optimal stopping times is then given by:

$$\begin{cases} \tau_t^{(1)} = \inf \left\{ k > t \mid I_{t|\emptyset}^{(1)} = 1 \right\}. \\ \tau_t^{(i)} = \inf \left\{ \tau_t^{(i-1)} < k \mid I_{t|\{\tau_t^{(1)}, \dots, \tau_t^{(i-1)}\}} = 1 \right\}, \\ i = 2, \dots, n. \end{cases} \quad (47)$$

Let $(\tau)_{t,n} = (\tau_t^{(1)}, \dots, \tau_t^{(n)})$. The real option fair value at time t is then given by:

$$V_t^{(n)} = \mathbb{E}_t \left[Z_{\{(\tau)_{t,n}\}}^d \right]. \quad (48)$$

Replacing the different expected rewards by their counterparts which are obtained via the LSM approach yields a (sub)optimal n -compound exercise policy $(\tilde{\tau})_{t,n}$ and a lower bound $\underline{V}_t^{(n)}$ can be obtained by:

$$\underline{V}_t^{(n)} = \mathbb{E}_t \left[Z_{\{(\tilde{\tau})_{t,n}\}}^d \right]. \quad (49)$$

4 The multiple exercise contract: back to a Markovian setting

The compound stopping approach allows to overcome the coupling between exercising and the change in the terminal duration and the contract maturity by consequence. This approach can, however, be very time consuming because of the complexity of the embedded dynamic programming problems and the time that they require for computation especially for long duration contracts and contracts with a high number of exercise rights. This section presents another approach to tackle the contract's valuation. The proposed approach simplifies the decision problem to a Markovian setting. The main limitation in the valuation of the MRG-LPV is the immediate change in the contract duration, after exercising. The decision maker has then to remember the dates at which all the previous rights were exercised in order to know the date at which the contract will end and her compensation, by consequence. To overcome this limitation, one can introduce an additional parameter in the problem characterization. The decision maker can have a perceived duration T_t of the terminal date of the contract at each decision time $t < T_{max}$ which can be defined as follows:

$$T_t = \begin{cases} \min \left\{ T_{max}, \inf \left\{ k \mid \mathbb{E}_t [PVR_k] \geq PVR_0 \right\} \right\}, \\ \text{if } t < T^{(*)}, \\ T^{(*)} & \text{otherwise.} \end{cases} \quad (50)$$

The perceived duration changes immediately after a payment H_t is made. Let $T_t(H_t)$ be the new perceived duration after a certain payment H_t is received and it is defined as follows:

$$T_t(H_t) = \begin{cases} \min \left\{ T_{max}, \inf \{k \mid H_t + \right. \\ \left. \mathbb{E}_t [PVR_k] \geq PVR_0 \} \right\}, & \text{if } t < T^*(H_t), \\ T^*(H_t) & \text{otherwise.} \end{cases} \quad (51)$$

Including the perceived duration as an additional parameter in the definition of the state at a given decision time should simplify the valuation problem as the following development intends to show.

Let $V_{t|T_t}^{(i)}$, $i = 1, \dots, n$ denote the perceived value of the contract with i rights when the decision maker sees at t a perceived duration T_t . Let $T_t^{(i-1)}$, $i = 1, \dots, n$ denote the perceived duration when there are $i - 1$ rights which have been already claimed. One have $T_t^{(0)} = T_t$. The decision maker problem, if there are n rights yet to be claimed, can be summarized as follows:

- Exercise and claim a compensation of Z_t^d which does not require any prior knowledge to determine. Her action changes then the contract's perceived duration from T_t to $T_t^{(1)}$. She holds, afterwards, a contract with $n - 1$ remaining rights and a new perceived maturity $T_t^{(1)}$. The next exercise can be made starting from the next time period $t + 1$. Her perceived reward is, therefore, given by the following continuation value:

$$Q_{t|T_t}^{(n-1)} = Z_t^d + \mathbb{E}_t \left[V_{t+1|T_t^{(1)}}^{(n-1)} \right]. \quad (52)$$

- continue. In that case, the contract duration remains T_t and the decision maker holds the same contract where she can exercise starting from the next time $t + 1$. Her perceived reward is given by:

$$Q_{t|T_t}^{(n)} = \mathbb{E}_t \left[V_{t+1|T_t}^{(n)} \right] \quad (53)$$

$Q_{t|T_t}^{(n)}$ can be naturally derived from the current dynamic programming. The main difficulty for this approach is the determination of $\mathbb{E}_t \left[V_{t+1|T_t^{(1)}}^{(n-1)} \right]$ which would naturally require to start the valuation procedure for the $(n - 1)$ -right contract with the new perceived maturity. This may be very time consuming and should revert to the compound stopping approach. Since the aim of this section is to present a faster approach to estimate the contract's value, an additional simplification has to be made to construct a computationally effective valuation procedure. We have so far extended the definition of the state at a time t from $S_t = (R_t, \xi_t)$ to $S'_t = (R_t, \xi_t, T_t)$. The new current state can be used, in the regression procedure,

to derive an approximation of $Q_{t|T_t}^{(n-1)}$. In other words, an approximation of $Q_{t|T_t}^{(n-1)}$ can be simply derived by adding a set of basis functions which depend on the perceived duration T_t . The difficulty that the immediate change leads to fades since the continuation value is directly parametrized by the perceived duration. The gain in time in comparison with the compound stopping approach is sizable (see results in table 3). The perceived values of the contract are, therefore, given by the means of the following Bellman equation:

$$\begin{cases} V_{T_{max}}^{(n)} = Z_{T_{max}}^d, \\ V_{t+1}^{(n)} = \max \left(Z_t + \mathbb{E}_t \left[V_{t+1|T_t^{(1)}}^{(n-1)} \right], \mathbb{E}_t \left[V_{t+1|T_t}^{(n)} \right] \right). \end{cases} \quad (54)$$

An exercise indicator for the i^{th} right can be determined as follows:

$$I_t^{(i)} = \begin{cases} 1 & \text{if } Q_{t|T_t}^{(i-1)} \geq Q_{t|T_t}^{(i)} \\ 0 & \text{otherwise,} \end{cases} \quad (55)$$

The stopping time for the n^{th} right (which is exercised first) can be then obtained as follows:

$$\tau_t^{(n)} = \inf \left\{ k \mid I_k^{(n)} = 1 \right\} \quad (56)$$

The near optimal stopping time for the i^{th} , $i = n - 1, \dots, 1$ right can be afterwards obtained as follows:

$$\tau_t^{(i)} = \inf \left\{ \tau_t^{(i+1)} < k \mid I_k^{(i)} = 1 \right\} \quad (57)$$

The procedure requires a constant update of the perceived duration which is defined as follows:

$$T_k^{(i)} = T_k \left(\sum_{j=1}^{i-1} H_{\tau_t^{(j)}} \right), \quad k = \tau_t^{(i-1)}, \dots, T_{max}. \quad (58)$$

The use of this recursive procedure allows then to derive an approximation of the contract's value by replacing the different continuation values by their approximations (e.g by the LSM approach).

5 Numerical illustration

This section aims to illustrate the impact of the MRG-LPVR on the financial viability of a PPP transaction. For this purpose, we consider a project for which the construction duration is estimated at 3 years with a total construction cost of 150 Million € expressed in present value. The construction costs inflation is assumed at 1% for the upcoming three years. The SPV is funded by equity (15%) and non-recourse debt (85%). The requested level of equity is a constraint that the

public entity imposes on private operators to guarantee a minimal involvement in the project. The lenders grant the SPV a grace period during the construction of the project where she does not reimburse capital. Once the construction is over, the interest on the borrowed capital is of 7.5%. The debt's maturity is of 25 years starting from the construction termination. The expected return on equity is 10%. The risk free rate is 4%. After the termination of the construction, the project requires an annual cost of 3.5 Million € for operation and maintenance. The inflation of this cost is estimated at 3%. The tax rate is 33%. The project's revenue is assumed to follow a Geometric Brownian motion:

$$dR_t = \mu R_t dt + \sigma R_t dW_t, \quad (59)$$

where $\mu = 2\%$ is the annual expected revenue increment, $\sigma = 15\%$ is the revenue volatility and W_t is a Wiener process. The initial value of the revenue is estimated at 15 Million €. The maximal duration for the contract is set at $T_{max} = 50$ years and the discounting rate for the Present value of revenue is equal to the weighted average cost of capital $r_c = 7.45\%$. The analysis focuses on the following financial indicators ⁷:

- $\mathbb{E}[NPV]$: the expected net present value of the project's sponsors. It is governed by the dividends that the SPV generates after she meets all its legal obligations,
- $\mathbb{E}[NPV^g]$: the expected net present value of the public entity. It is mainly governed by the cash flows caused by the guarantee and the generated cash flows once the project is transferred. The discounting is made via the risk-free rate,
- $\mathbb{E}[\mathbb{E}[DSCR_t]]$: the expected value of the average Debt Service Coverage Ratio over the debt's life. The project's bankability is an increasing function of $\mathbb{E}[\mathbb{E}[DSCR_t]]$,
- $\mathbb{E}[T^*]$ the average duration of the contract,
- $p^* = \mathbb{P}[PVR^* < PVR_0]$: the probability that the targeted present value of revenue is not collected at the termination of the contract.

The public decision maker starts by analyzing the financial viability of the project in the absence of the guarantee and for different levels of target Present value of revenue PVR_0 as shown in figures 2, 3, 4 which present respectively the NPV , $\mathbb{E}[T^*]$ and p^* for different levels of targeted PVR_0 . The number of simulations is 10 000 in the whole presentation.

The initial analysis shows that the contract is not financially viable because of the negative NPV that it leads to and figure 4 shows that the chance of not collecting PVR_0 is considerable. This is mainly due to the high variability of the revenue and its low level. The public entity can provide

⁷For a detailed analysis on discounted cash flow analysis in Public Private Partnerships, one can refer to (Zhang, 2005; M.M.Islam, 2008).

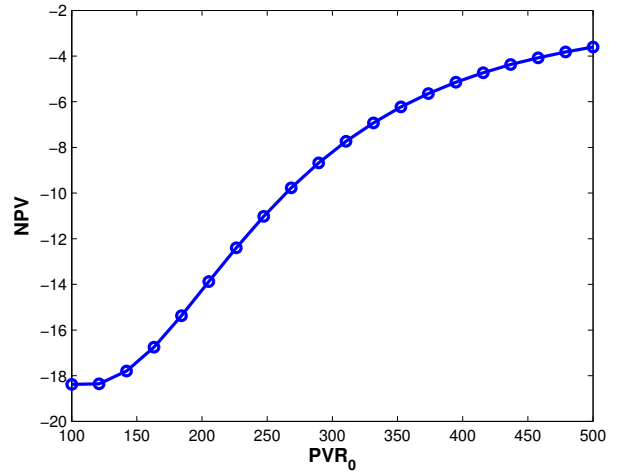


Figure 2: Comparing the NPV for different levels of PVR_0 .

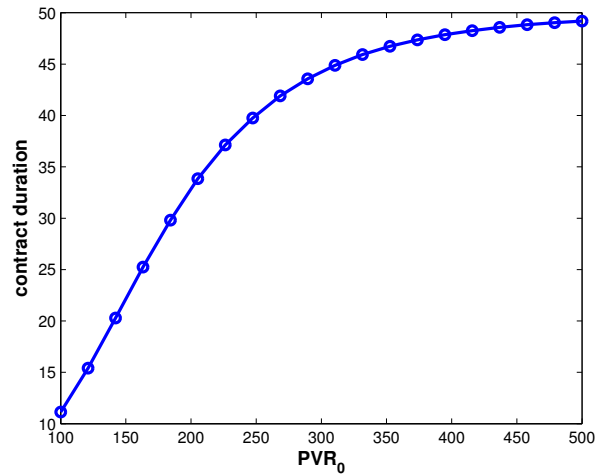


Figure 3: Comparing contract expected duration for different levels of PVR_0 .

an initial subsidy for the project in order to guarantee its financial viability, but this would be a certain expense. It may opt, however, for a flexible contract duration with some contingent subsidies presented in the form of MRG-LPVR. It targets, moreover, an expected duration that ranges between 30 to 35 years. It focuses therefore the analysis on three levels of $PVR_0 = 200, 225, 250$. To value the financial viability of the project, in the presence of the guarantee, the following steps should be followed:

1. make a stochastic discounted cash flow analysis of the project,
2. estimate the volatility of the logarithmic return of the project and compute the stochastic discounting factor

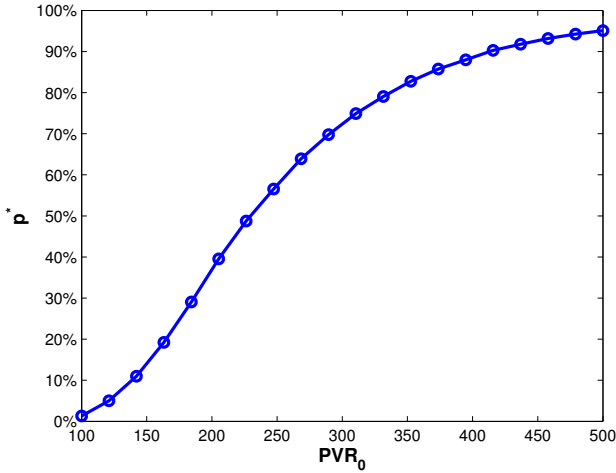


Figure 4: Comparing p^* for different levels of PVR_0 .

as in equations (2) and (8). For the determination of the conditional expectation, we use the first 4 Laguerre polynomials ,

3. compute the value of the guarantee following the steps in section 4. The set of basis function for the LSM approach is enlarged by the SDF ξ_t , the payoff $Z_t^{(1)}$ and the perceived duration T_t ,
4. Analyse the impact of the guarantee on the financial indicator of the project by determining the near optimal stopping policy and re-adjusting the cash flows.

Table 1 summarizes the value of the guarantee for some exercise rights and the different PVR_0 . Figure 5 presents the impact of the MRG-LPVR on the net present value of equity. It shows the enhancement of the project's return as the number of exercise rights n grows. In figure 6, one can see the opposite effect on the public NPV. The MRG contract allows also to boost the project's bankability as presented in 7. Moreover, the MRG-LPVR reduces p^* as well as the expected duration of the contract as presented respectively in figures 8 and 9. Figure 10 presents a comparison between the values of the flexible and static contracts.

The previous analysis presents one of the major pitfalls of the Least Present Value of revenue scheme. In fact, in environments with high variability of revenue, there is a substantial chance for the private operator not be able to recoup the targeted PVR. Increasing PVR_0 would improve the outcome of the project in expectation, however the chance of not collecting PVR_0 increases. There is therefore a certain trade-off between PVR_0 (and the expected return in consequence) and the chance of not collecting the targeted value of revenue. In other words, the outcome of the project increases by increasing PVR_0 , and so does its variability. Figure 11 illustrates

n	PVR_0		
	200	225	250
1	3.37	3.55	3.67
2	6.69	7.04	7.28
3	10.15	10.55	10.85
4	13.35	13.86	14.29
10	29.71	31.85	33.23
20	50.62	57.00	60.37
30	61.50	76.01	82.17
35	62.11	82.89	90.83

Table 1: Comparing the guarantee value with different exercise numbers and targeted present value of revenue

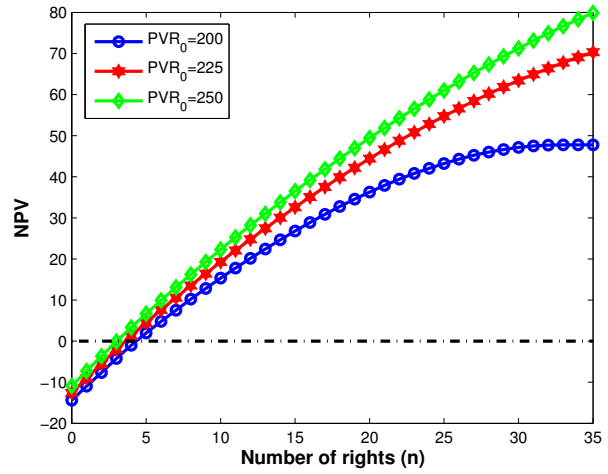


Figure 5: Impact of the guarantee on the Net present value of the project.

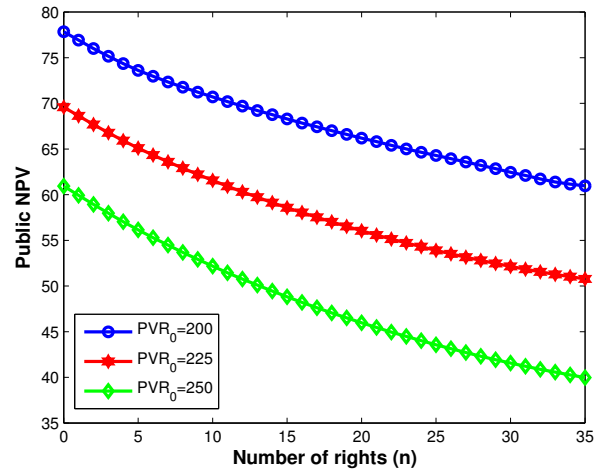


Figure 6: Impact of the guarantee on the public Net present value of the project.

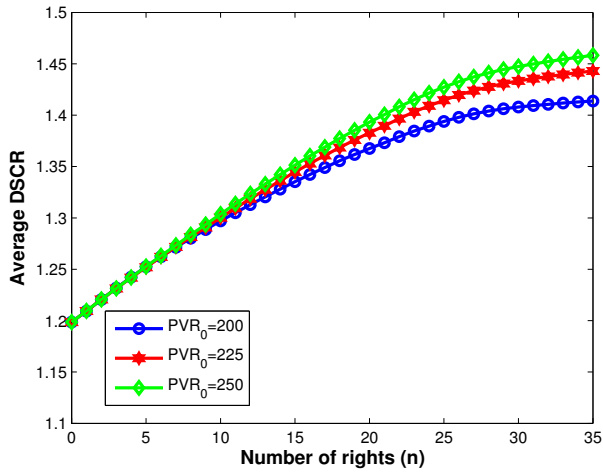


Figure 7: Impact of the guarantee on the Average Debt Coverage Ratio.

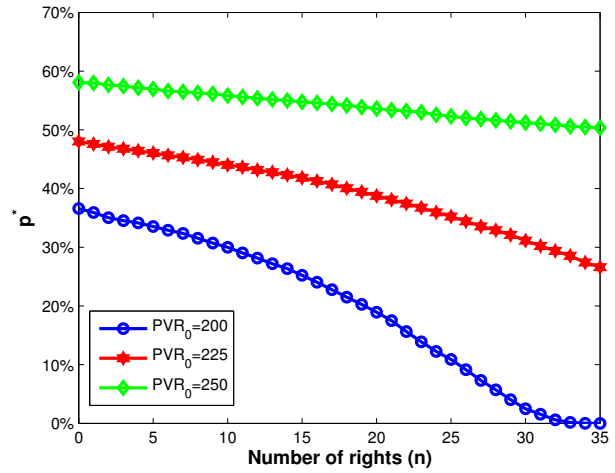


Figure 9: Impact of the guarantee on the probability of not recouping the target present value of revenue.

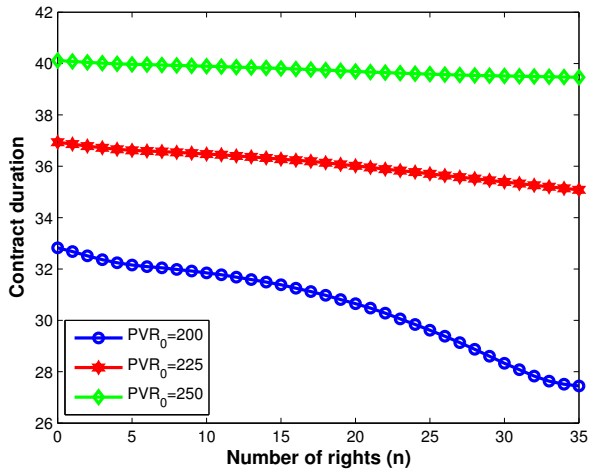


Figure 8: Impact of the guarantee on the contract's duration.

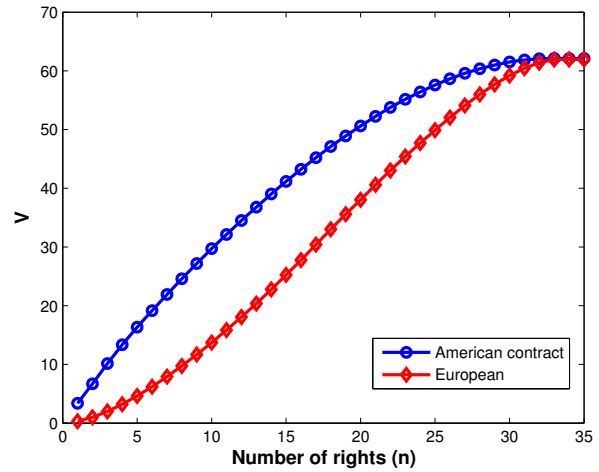


Figure 10: Comparing the American and the European guarantee.

this effect and presents the expected percentage of the collected PVR^* with respect to the initial requested PVR^0 . Introducing the guarantee can help mitigate this pitfall and reduce the private risk. It allows, moreover, to reduce the targeted PVR_0 . One can see in figure 5 that with $n = 5$, the expected NPV is positive and this for the different PVR_0 , however the risk is reduced since the chance of not collecting the targeted PVR_0 is respectively 34%, 46% and 57%, as shown in figure 9. The overall cost to the public entity is also reduced as shown in figure 6 because of the lower PVR_0 and the reduction of the contract's duration (figure 8). This effect is better shown in figure 12, where one can see that at a constant expected cost for the public entity, p^* can be con-

siderably reduced for lower PVR_0 . Figure 13 indicates that a reasonable private NPV can be reached with a lower cost for the public entity for the lowest PVR_0 . These different effects are due on one hand to the reduction of the volatility of the project's return as PVR_0 decreases as presented in figure 14. On the other hand, the introduction of the guarantee reduces the payback period of the project, since higher cash flows come at earlier stages of the project.

All in all, one can argue that the introduction of the guarantee increases the return of the project and reduces the project's risk for the private operator. It makes the structuring of the project with lower PVR_0 possible which may increase the social present value of the project in the long-run.

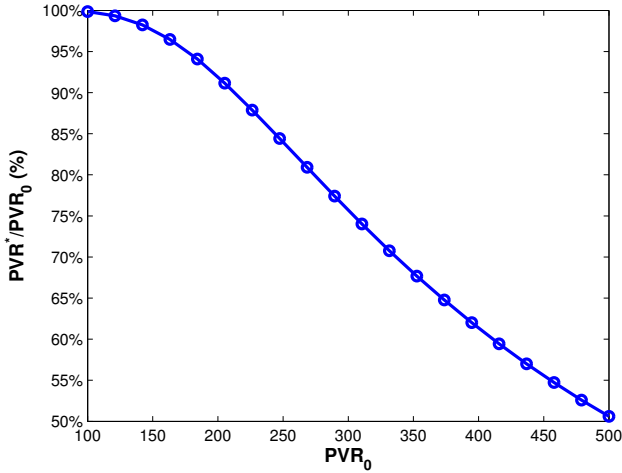


Figure 11: Expected recouped present value of revenue as a percentage of the initial required present value of revenue

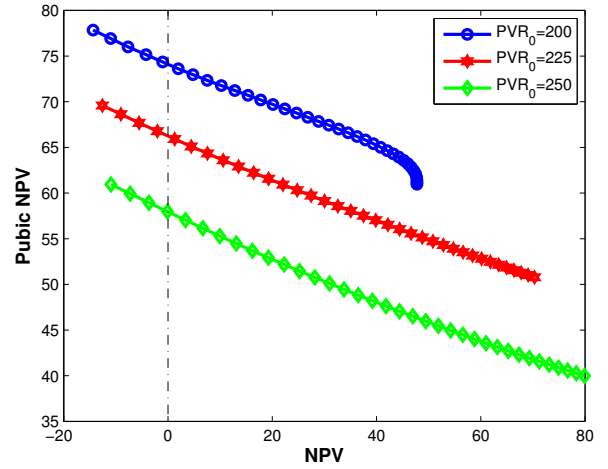


Figure 13: Polar representation of the public and private NPV

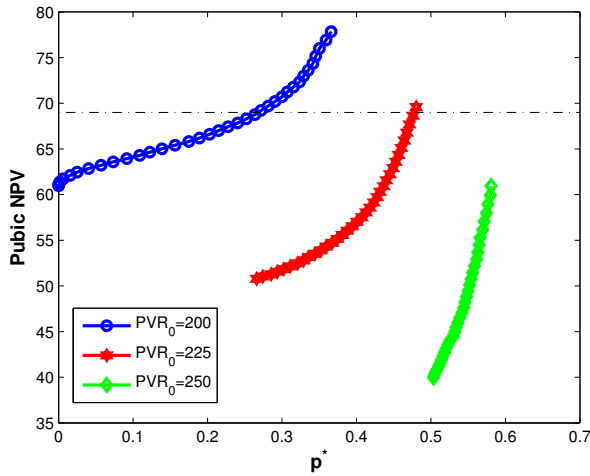


Figure 12: Polar representation of p^* and the public NPV

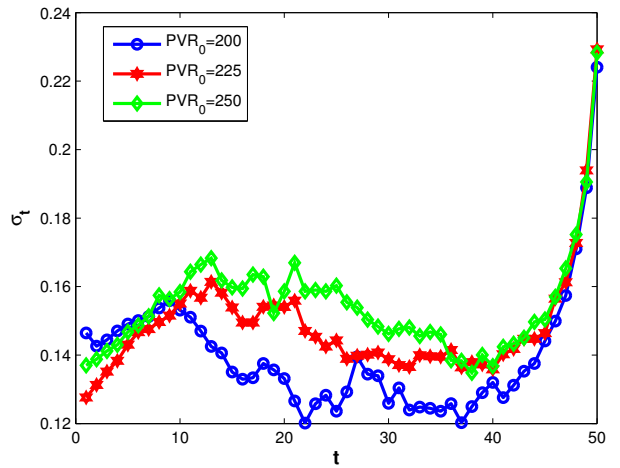


Figure 14: Project's volatility over time for different targeted Present Values of Revenue

Table 2 presents the relative difference δV between the compound stopping approach and the simplified approach defined as follows:

$$\delta V = \frac{V^c - V^s}{V^c}, \quad (60)$$

where V^c and V^s denote respectively the value obtained by the compound stopping and simplified procedures. The results are not conclusive with a slight advantage for the simplified approach for higher exercise rights. The compound stopping approach does not require the approximation of the duration of the contract and has an advantage in this regard in comparison with the simplified approach. However, there is a higher number of conditional expectation to approximate and the lower bias of the approximation may increase. This bias

will naturally increase for higher number of exercise rights. Table 3 presents the relative difference $\delta \theta$ in computational times:

$$\delta \theta = \frac{\theta^c - \theta^s}{\theta^c}, \quad (61)$$

where θ^c and θ^s denote respectively the computational time for the compound stopping and simplified procedures. One can clearly see the enormous gain in time that the simplification of the problem induces. This is mainly due to the fact that the compound stopping approach requires the exploration of the set of all possible combinations of stopping times which is not the case for the simplified approach. In addition, there is a higher number of conditional expectations to approximate. Another advantage of the simplified problem is the recursivity

of the valuation algorithm. In other words, if one would value the contract with n rights, he can automatically have access to the values of the contracts with 1 to $n - 1$ rights which is not the case for the compound stopping approach.

n	PVR ₀		
	200	225	250
$\sigma = 10\%$			
2	-0,01	0,81	0,36
3	-1,92	-0,62	-0,84
$\sigma = 15\%$			
2	0,50	0,33	0,55
3	-2,11	0,35	-0,47
$\sigma = 20\%$			
2	0,40	0,21	-0,08
3	-2,08	-1,45	-0,85

Table 2: Relative difference between values obtained by the compound stopping and the simplified valuation procedures (values are in percentage).

n	PVR ₀		
	200	225	250
$\sigma = 10\%$			
2	67	68	72
3	2090	1811	1628
$\sigma = 15\%$			
2	81	92	106
3	1946	2548	1761
$\sigma = 20\%$			
2	155	101	155
3	1614	2125	1614

Table 3: Relative difference between computational times of the compound stopping and the simplified valuation procedures (values are in percentage)

6 Conclusion

The design of flexible public private partnerships is essential to increase their appeal for private bidders and to boost the cooperation between the private and the public sector during the whole life of the project. Following this spirit, this work presents a novel guarantee which aims to boost the financial viability of PPP projects with variable durations. The contingent claim is presented as a multiple exercise American option with floating maturity. The valuation is made under the real world measure by introducing a stochastic discounting factor which guarantees the market-consistency of the valuation procedure and guarantees a better management of risk. The valuation is first considered as a compound stopping problem and is later simplified to a Markovian setting. The latter approach permits to considerably reduces the computational time that the valuation procedure requires. Our numerical results show the substantial effect of the guarantee in improving the financial viability of PPP project and in guaranteeing a better risk

sharing between the stakeholders. It shows also that the guarantee may reduce the overall cost to the public entity while increasing the project's return and reducing the private risk.

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